BENG280A, Principles of Biomedical Imaging 10/11/04
Fall Quarter 2004

Fourier Transform of comb(x)

In class, we stated without proof that the Fourier transform of *comb(x)* is *comb(k_x)*. There are number of ways to motivate and demonstrate this result [see references below]. The derivation here is similar to that in references 2 and 3. Since *comb(x)* is a periodic "function" with period $X = 1$, we can think of expanding it as a Fourier series

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{m=-\infty}^{\infty} c_m e^{j2\pi mx/X} = \sum_{m=-\infty}^{\infty} c_m e^{j2\pi mx}$$

with expansion coefficients

$$\begin{aligned} c_m &= \left\langle e^{j2\pi mx}, comb(x) \right\rangle \\ &= \int_{-1/2}^{1/2} \sum_{n=-\infty}^{\infty} \delta(x-n) e^{-j2\pi mx} dx \\ &= \int_{-1/2}^{1/2} \delta(x) e^{-j2\pi mx} dx \\ &= 1 \end{aligned}$$

Thus, the Fourier series expansion is $comb(x) = \sum_{m=-\infty}^{\infty} e^{j2\pi mx} = 2\sum_{m=0}^{\infty} \cos(2\pi mx)$. At this point, it might be useful to sketch out what this infinite series looks like, and convince yourself that it is plausible for the sum to yield a comb function. We can now take the Fourier Transform of the Fourier series expansion to obtain

$$\begin{aligned} F\left[comb(x)\right] &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{j2\pi mx} e^{-j2\pi k_x x} dx \\ &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi mx} e^{-j2\pi k_x x} dx \\ &= \sum_{m=-\infty}^{\infty} \delta(k_x - m) \end{aligned}$$

[1] Ronald Bracewell, *The Fourier Transform and Its Applications*. McGraw-Hill, New York 1986.
[2] Robert M. Gray and Joseph W. Goodman, *Fourier Transforms: An Introduction for Engineers*. Kluwer 1995.
[3] Martin Vetterli and Jelen Kovacevic, *Wavelets and Subband Coding*. Prentice-Hall, Englewood Cliffs, New Jersey, 1995.