Solution to Homework 3, Problem 3b

Multiplication in the Fourier domain results in convolution in the image domain so that:

$$F^{-1}\left(M(k_x,k_y)\sum_{n=-\infty}^{\infty}\delta\left(k_y-\frac{2n+1}{4L}\right)\right)=m(x,y)*F^{-1}\left(\sum_{n=-\infty}^{\infty}\delta\left(k_y-\frac{2n+1}{4L}\right)\right).$$

We use the properties of the delta function and the definition of the comb function to write:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\delta\left(k_y-\frac{2n+1}{4L}\right)&=\sum_{n=-\infty}^{\infty}\delta\left(\frac{1}{2L}\left(2Lk_y-\frac{1}{2}-n\right)\right)\\&=2L\sum_{n=-\infty}^{\infty}\delta\left(2Lk_y-\frac{1}{2}-n\right)\\&=2Lcomb(2Lk_y-1/2)\end{aligned}$$

In class we derived the result that

$$F^{-1}\left(G(ak_x+b)\right)=\frac{1}{|a|}g\left(\frac{x}{a}\right)e^{-j2\pi xb/a}\ .$$

With $a=2L$ and $b=-1/2$, we may write

$$\begin{aligned}F^{-1}\left(2Lcomb(2Lk_y-1/2)\right)&=comb\left(\frac{y}{2L}\right)\exp\left(\frac{j\pi y}{2L}\right)\\&=\sum_{n=-\infty}^{\infty}\delta\left(\frac{y}{2L}-n\right)\exp\left(\frac{j\pi y}{2L}\right)\\&=\sum_{n=-\infty}^{\infty}\delta\left(\frac{1}{2L}\left(y-n2L\right)\right)\exp\left(\frac{j\pi y}{2L}\right)\\&=2L\sum_{n=-\infty}^{\infty}\delta\left(y-n2L\right)\exp\left(\frac{j\pi y}{2L}\right)\\&=2L\sum_{n=-\infty}^{\infty}\delta\left(y-n2L\right)\exp\left(j\pi n\right)\\&=2L\sum_{n=-\infty}^{\infty}\delta\left(y-n2L\right)(-1)^n\end{aligned}$$

This is just a modulated train of delta functions with spacing of *2L*, with amplitudes of *2L* for *n* even and amplitudes of *–2L* for *n* odd. Since there was no k_x dependence in the sampling function, the full 2D inverse Fourier transform is $2L\delta(x)\sum_{n=-\infty}^{\infty}\delta\left(y-n2L\right)(-1)^n$. Convolving this with $m(x,y)=\delta(x)\left(\delta(y-L)+\delta(y+L)\right)$ yields $2L\delta(x)\sum_{n=-\infty}^{\infty}\left[\delta\left(y-n2L-L\right)-\delta\left(y-n2L+L\right)\right](-1)^n$. In class, we showed graphically that this could be written as $4L\delta(x)\sum_{n=-\infty}^{\infty}\left[\delta\left(y-n2L-L\right)\right](-1)^{n}$. To show this in a more rigorous fashion, we can split the sum into two infinite sums and use the substitution $n'=n+1$ to write

$$\begin{aligned}
2L\delta(x)\sum_{n=-\infty}^{\infty}\left[\delta\left(y-n2L-L\right)-\delta\left(y-n2L+L\right)\right](-1)^{n} &= 2L\delta(x)\left(\sum_{n=-\infty}^{\infty}\delta\left(y-n2L-L\right)(-1)^{n}-\sum_{n=-\infty}^{\infty}\delta\left(y-n2L+L\right)(-1)^{n}\right)\\
&= 2L\delta(x)\left(\sum_{n=-\infty}^{\infty}\delta\left(y-n2L-L\right)(-1)^{n}+\sum_{n'=-\infty}^{\infty}\delta\left(y-n'2L-L\right)(-1)^{n'}\right)\\
&= 4L\delta(x)\sum_{n=-\infty}^{\infty}\delta\left(y-n2L-L\right)(-1)^{n}
\end{aligned}$$