Bioengineering 280A
Principles of Biomedical Imaging
Fall Quarter 2004
MRI Lecture 2

Today’s Topics

• Bloch Equation
• Gradients
• Signal Equation
• k-space trajectories
• Spin-echo/ gradient echo

Magnetic Moment and
Angular Momentum

A charged sphere spinning about its axis has angular momentum and a magnetic moment.

This is a classical analogy that is useful for understanding quantum spin, but remember that it is only an analogy!

Relation: \( \mathbf{\mu} = \gamma \mathbf{S} \) where \( \gamma \) is the gyromagnetic ratio and \( \mathbf{S} \) is the spin angular momentum.
Precession

\[ \frac{d\mu}{dt} = \mu \times \gamma B \]

Analogous to motion of a gyroscope

Precesses at an angular frequency of

\[ \omega = \gamma B \]

This is known as the Larmor frequency.

Magnetization Vector

\[ M = \frac{1}{V} \sum_{\text{protons in } V} \mu_i \]

Vector sum of the magnetic moments over a volume.

For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

Equation of motion is the same form as for individual moments.

RF Excitation

\[ M_0 (1 - e^{-t/T_1}) \]

T1 recovery

\[ e^{t/T_2} \]

T2 decay
Bloch Equation

\[
d\mathbf{M}/dt = \mathbf{M} \times \gamma \mathbf{B} - \frac{M_i \mathbf{i} + M_j \mathbf{j}}{T_2} - \frac{(M_z - M_0) \mathbf{k}}{T_1}
\]

- Precession
- Transverse Relaxation
- Longitudinal Relaxation

\(\mathbf{i}, \mathbf{j}, \mathbf{k}\) are unit vectors in the x,y,z directions.

Free precession about static field

\[
d\mathbf{M}/dt = \mathbf{M} \times \gamma \mathbf{B}
\]

\[
= \gamma \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\begin{bmatrix}
M_x & M_y & M_z \\
B_x & B_y & B_z
\end{bmatrix}
\]

\[
= \gamma \begin{bmatrix}
B_x(M_y - B_y M_z) \\
B_y(M_x - B_x M_z) \\
B_z(M_x - B_x M_y)
\end{bmatrix}
\]

Free precession about static field

\[
\begin{bmatrix}
dM_x/dt \\
dM_y/dt \\
dM_z/dt
\end{bmatrix}
= \gamma \begin{bmatrix}
B_x (M_y - B_y M_z) \\
B_y (M_x - B_x M_z) \\
B_z (M_x - B_x M_y)
\end{bmatrix}
\]

\[
= \gamma \begin{bmatrix}
0 & B_y & -B_z \\
-B_x & 0 & B_z \\
B_z & -B_y & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]
Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & B_0 & 0 \\
-B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define \( M = M_x + jM_y \)

\[
\frac{dM}{dt} = \frac{d}{dt}(M_x + iM_y)
\]
\[
= -jB_0 M
\]

Solution is a time-varying phaser

\[
M(t) = M(0)e^{-j\omega_0 t} = M(0)e^{-j\omega_0 t}
\]

In matrix form this is

\[
\begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} =
\begin{bmatrix}
\cos \omega_0 t & \sin \omega_0 t & M_x(0) \\
-\sin \omega_0 t & \cos \omega_0 t & M_y(0) \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
M_x(0) \\
M_y(0) \\
M_z(0)
\end{bmatrix}
\]

The full solution is then a rotation about the z-axis.

Matrix Form with \( B = B_0 \)

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} =
\begin{bmatrix}
-1/T_2 & yB_0 & 0 \\
-yB_0 & 1/T_2 & 0 \\
0 & 0 & -1/T_2
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
M_x(t) = M_x(0)e^{-j\omega_0 t} = M_x(0)e^{-j\omega_0 t}
\]

\[
M_y(t) = M_y(0)e^{-j\omega_0 t} = M_y(0)e^{-j\omega_0 t}
\]

\[
M_z(t) = M_z(0)e^{-j\omega_0 t} = M_z(0)e^{-j\omega_0 t}
\]
Z-component solution

\[ M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1} \]

Saturation Recovery
If \( M_z(0) = 0 \) then \( M_z(t) = M_0(1 - e^{-t/T_1}) \)

Inversion Recovery
If \( M_z(0) = -M_0 \) then \( M_z(t) = M_0(1 - 2e^{-t/T_1}) \)

Transverse Component

\[ M = M_x + jM_y \]

\[ \frac{dM}{dt} = \frac{d}{dt}(M_x + jM_y) = -j(\omega_0 + 1/T_2)M \]

\[ M(t) = M(0)e^{-\omega_0 t}e^{-t/T_2} \]

Summary
1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

Fact: Can show that \( T_2 < T_1 \) in order for \( |M(t)| \leq M_0 \).
Physically, the mechanisms that give rise to \( T_1 \) relaxation also contribute to transverse \( T_2 \) relaxation.
Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to $B_z$ such that $B_z(x,y,z) = B_0 + \Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.

MRI System

Simplified Drawing of Basic Instrumentation. Body lies on table encompassed by coils for static field $B_0$, gradient fields (two of three shown), and radiofrequency field $B_1$.

Z Gradient Coil

B(mT) vs. L
Gradient Fields

\[ B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z \]

\[ = B_0 + G_x x + G_y y + G_z z \]

\[ G_x = \frac{\partial B_z}{\partial z} > 0 \quad \quad G_y = \frac{\partial B_z}{\partial y} > 0 \]

Define

\[ \vec{G} = G_x \hat{i} + G_y \hat{j} + G_z \hat{k} \]

So that

\[ G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r} \]

Also, let the gradient fields be a function of time. Then the \( z \)-directed magnetic field at each point in the volume is given by:

\[ B_z(\vec{r},t) = B_0 + \vec{G}(t) \cdot \vec{r} \]

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

\[ M(t) = M(0) e^{-j\omega_0 t} e^{-j/2} \]

In the presence of non time-varying gradients we have

\[ M(\vec{r}) = M(\vec{r},0) e^{-jB_z(\vec{r}) \hat{z} e^{-j/2} \vec{r}} = M(\vec{r},0) e^{-jB_0 + \vec{G}(t) \cdot \vec{r}} e^{-j/2} \vec{r} = M(\vec{r},0) e^{-jB_0 + \vec{G}(t) \cdot \vec{r}} e^{-j/2} \vec{r} = M(\vec{r},0) e^{-jB_0 + \vec{G}(t) \cdot \vec{r}} e^{-j/2} \vec{r} \]

\[ = M(\vec{r},0) e^{-jB_0 + \vec{G}(t) \cdot \vec{r}} e^{-j/2} \vec{r} \]
Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

\[ \omega(\vec{r}, t) = \gamma B_0 + \gamma \tilde{G}(t) \cdot \vec{r} \]

\[ = \omega_0 + \Delta \omega(\vec{r}, t) \]

The phase of each spin is

\[ \phi(\vec{r}, t) = -\int_0^t \omega(\vec{r}, \tau) d\tau \]

\[ = -\omega_0 t + \Delta \phi(\vec{r}, t) \]

Where the incremental phase due to the gradients is

\[ \Delta \phi(\vec{r}, t) = -\int_0^t \Delta \omega(\vec{r}, \tau) d\tau \]

\[ = -\int_0^t \gamma \tilde{G}(\tau) \cdot \vec{r} d\tau \]

The transverse magnetization is then given by

\[ M(\vec{r}, t) = M(\vec{r}, 0) e^{-j/2} e^{j \omega(\vec{r}, t) t} \]

\[ = M(\vec{r}, 0) e^{-j/2} e^{j \omega(\vec{r}, 0) t} \exp \left( -j \int_0^t \Delta \omega(\vec{r}, \tau) d\tau \right) \]

\[ = M(\vec{r}, 0) e^{-j/2} e^{j \omega_0 t} \exp \left( -j \int_0^t \gamma \tilde{G}(\tau) \cdot \vec{r} d\tau \right) \]

It can be shown that this satisfies the differential equation

\[ dM / dt = d / dt \left( M_1 + i M_2 \right) \]

\[ = -j(\omega_0 + 1/T_2) M \]
Signal Equation

Signal from a volume
\[ s(t) = \int M(r,t) dV - \int \int M(x,y,z,0) e^{-\gamma G_x(t) r \cdot \hat{r}} \exp \left[ -j \omega_0 t \right] \exp \left[ -j \gamma G_y(t) r \cdot \hat{r} \right] \int dx \int dy \int dz \]

For now, consider signal from a slice along z and drop the \( T_2 \) term. Define
\[ m(x,y) \equiv M(x,y,z) \left[ z_0 - \frac{\Delta z}{2} \right] + \frac{\Delta z}{2} \int dz \]

To obtain
\[ s(t) = \int \int m(x,y) e^{-\gamma G_x(t) r \cdot \hat{r}} \exp \left[ -j \lambda_0 t \right] \exp \left[ -j \gamma G_y(t) r \cdot \hat{r} \right] \int dx \int dy \]

Demodulate the signal to obtain
\[ m(t) = e^{\gamma G_x(t)} \left[ \int m(x,y) e^{-\gamma G_x(t) r \cdot \hat{r}} \exp \left[ -j \lambda_0 t \right] \exp \left[ -j \gamma G_y(t) r \cdot \hat{r} \right] \int dx \int dy \right] \]

Where
\[ k_x(t) = \frac{1}{2 \pi} \int G_x(t) dt \]
\[ k_y(t) = \frac{1}{2 \pi} \int G_y(t) dt \]

MR signal is Fourier Transform
\[ s(t) = \int \int m(x,y) e^{-2 \pi \left[ k_x(t)x + k_y(t)y \right]} \exp \left[ -j \lambda_0 t \right] \exp \left[ -j \gamma G_y(t) r \cdot \hat{r} \right] \int dx \int dy = F \left[ m(x,y) \right]_{k_x(t),k_y(t)} \]
K-space

At each point in time, the received signal is the Fourier transform of the object
\[ s(t) = M(k_x(t), k_y(t)) = \mathcal{F}[m(x,y)] \]
evaluated at the spatial frequencies:
\[ k_x(t) = \frac{1}{2\pi} \int G_x(t) dt \]
\[ k_y(t) = \frac{1}{2\pi} \int G_y(t) dt \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

Interpretation

\[ \exp(-j \frac{2\pi}{\Delta x} \frac{1}{(2\pi)^2}) \]
\[ \exp(-j \frac{2\pi}{\Delta x} \frac{1}{(2\pi)^2}) \]
\[ \exp(-j \frac{2\pi}{\Delta x} \frac{1}{(2\pi)^2}) \]

\[ \Delta B(z) = G_x \]

Faster

Slower

K-space trajectory

\[ k_x(t) = \frac{1}{2\pi} \int G_x(t) dt \]
K-space trajectory

Spin-Warp