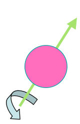
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
MRI Lecture 2

Today's Topics

- Bloch Equation
- Gradients
- Signal Equation
- k-space trajectories
- Spin-echo/ gradient echo

Magnetic Moment and Angular Momentum



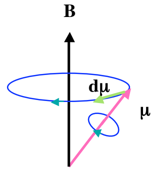
A charged sphere spinning about its axis has angular momentum and a magnetic moment.

This is a classical analogy that is useful for understanding quantum spin, but remember that it is only an analogy!

Relation: $\boldsymbol{\mu} = \gamma \mathbf{S}$ where γ is the gyromagnetic ratio and $\mathbf{S}$ is the spin angular momentum.

Precession

$$\frac{d\mu}{dt} = \mu \times \gamma \mathbf{B}$$



Analogous to motion of a gyroscope

Precesses at an angular frequency of

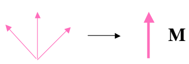
$$\omega = \gamma B$$

This is known as the **Larmor** frequency.

Magnetization Vector

$$\mathbf{M} = \frac{1}{V} \sum_{\substack{\text{protons} \\ \text{in V}}} \mu_i$$

Vector sum of the magnetic moments over a volume.

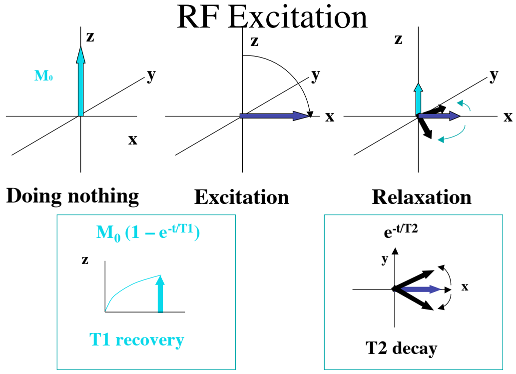


For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}$$

Equation of motion is the same form as for individual moments.

RF Excitation



Credit: Larry Frank

Bloch Equation

$$\frac{d\mathbf{M}}{dt} = \underbrace{\mathbf{M} \times \gamma \mathbf{B}}_{\text{Precession}} - \underbrace{\frac{M_x \mathbf{i} + M_y \mathbf{j}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0)\mathbf{k}}{T_1}}_{\text{Longitudinal Relaxation}}$$

i, j, k are unit vectors in the x,y,z directions.

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Free precession about static field

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}$$

$$= \gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \gamma \begin{pmatrix} \hat{i}(B_z M_y - B_y M_z) \\ -\hat{j}(B_z M_x - B_x M_z) \\ \hat{k}(B_y M_x - B_x M_y) \end{pmatrix}$$

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Free precession about static field

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} B_z M_y - B_y M_z \\ B_x M_z - B_z M_x \\ B_y M_x - B_x M_y \end{bmatrix}$$

$$= \gamma \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

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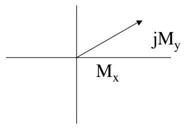
Precession

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_0 & 0 \\ -B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Useful to define $M \equiv M_x + jM_y$



$$dM/dt = d/dt\left(M_x + iM_y\right)$$
$$= -j\gamma B_0 M$$

Solution is a time-varying phasor

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

Precession

$$M(t) = M(0)e^{-j\omega_0 t}$$
$$= \left(M_x(0)\cos\omega_0 t + M_y(0)\sin\omega_0 t\right) + j\left(M_y(0)\cos\omega_0 t - M_x(0)\sin\omega_0 t\right)$$

In matrix form this is $\begin{bmatrix} M_x(t) \\ M_y(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \end{bmatrix}$

The full solution is then a rotation about the z-axis.

$$\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix}$$

$$= R_z\left(\omega_0 t\right) \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix}$$

Matrix Form with B=B$_0$

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \begin{bmatrix} -1/T_2 & \gamma B_0 & 0 \\ -\gamma B_0 & 1/T_2 & 0 \\ 0 & 0 & -1/T_1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0/T_1 \end{bmatrix}$$

Z-component solution

$$M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1}$$

Saturation Recovery

$$\text{If } M_z(0) = 0 \text{ then } M_z(t) = M_0(1 - e^{-t/T_1})$$

Inversion Recovery

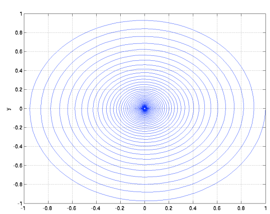
$$\text{If } M_z(0) = -M_0 \text{ then } M_z(t) = M_0(1 - 2e^{-t/T_1})$$

Transverse Component

$$M \equiv M_x + jM_y$$

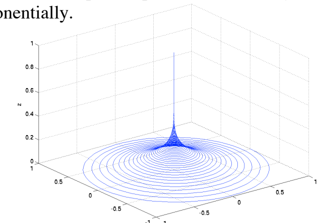
$$dM/dt = d/dt(M_x + iM_y) = -j(\omega_0 + 1/T_2)M$$

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$



Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.



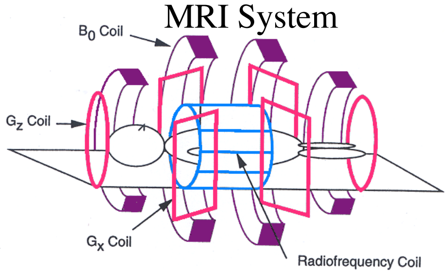
Fact: Can show that $T_2 < T_1$ in order for $|M(t)| \leq M_0$
Physically, the mechanisms that give rise to T_1 relaxation also contribute to transverse T_2 relaxation.

Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to B_z such that $B_z(x,y,z) = B_0 + \Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.

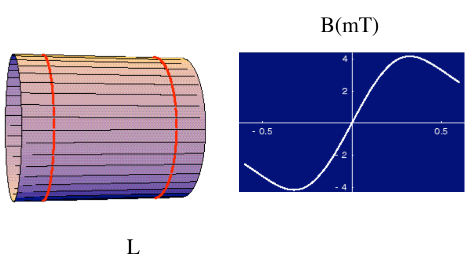
MRI System



Simplified Drawing of Basic Instrumentation.
Body lies on table encompassed by
coils for static field B_o,
gradient fields (two of three shown),
and radiofrequency field B_1.

Image, caption: copyright Nishimura, Fig. 3.15

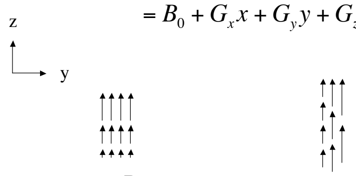
Z Gradient Coil



Credit: Buxton 2002

Gradient Fields

$$B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial x}x + \frac{\partial B_z}{\partial y}y + \frac{\partial B_z}{\partial z}z$$
$$= B_0 + G_x x + G_y y + G_z z$$



$$G_z = \frac{\partial B_z}{\partial z} > 0 \qquad G_y = \frac{\partial B_z}{\partial y} > 0$$

Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G}\cdot\vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t)\cdot\vec{r}$$

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

$$M(\vec{r}) = M(\vec{r},0)e^{-j\gamma B_z(\vec{r})}e^{-t/T_2(\vec{r})}$$
$$= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})}e^{-t/T_2(\vec{r})}$$
$$= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}}e^{-t/T_2(\vec{r})}$$

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned}\omega(\vec{r},t) &= \gamma B_z(\vec{r},t)\\ &= \gamma B_0 + \gamma \vec{G}(t)\cdot\vec{r}\\ &= \omega_0 + \Delta\omega(\vec{r},t)\end{aligned}$$

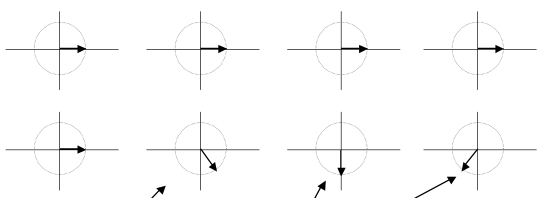
The phase of each spin is

$$\begin{aligned}\varphi(\vec{r},t) &= -\int_0^t \omega(\vec{r},\tau)d\tau\\ &= -\omega_0 t + \Delta\varphi(\vec{r},t)\end{aligned}$$

Where the incremental phase due to the gradients is

$$\begin{aligned}\Delta\varphi(\vec{r},t) &= -\int_0^t \Delta\omega(\vec{r},\tau)d\tau\\ &= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau\end{aligned}$$

Time-Varying Gradient Fields



Time-Varying Gradient Fields

The transverse magnetization is then given by

$$\begin{aligned}M(\vec{r},t) &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)}\\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right)\\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)\end{aligned}$$

It can be shown that this satisifies the differential equation

$$\begin{aligned}dM/dt &= d/dt(M_x + iM_y)\\ &= -j(\omega_0 + 1/T_2)M\end{aligned}$$

Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r},t)dV$$
$$= \int_x \int_y \int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x \int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$s(t) = e^{j\omega_0 t}s_r(t)$$
$$= \int_x \int_y m(x,y)\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$
$$= \int_x \int_y m(x,y)\exp\left(-j\gamma\int_o^t \left[G_x(\tau)x + G_y(\tau)y\right]d\tau\right)dxdy$$
$$= \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$s(t) = \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$
$$= M\left(k_x(t),k_y(t)\right)$$
$$= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

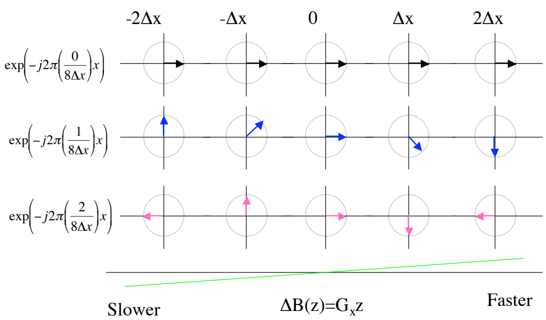
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

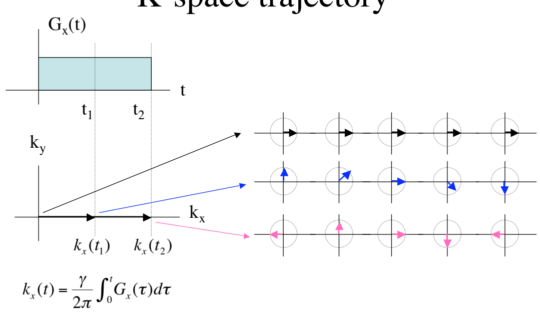
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

Interpretation

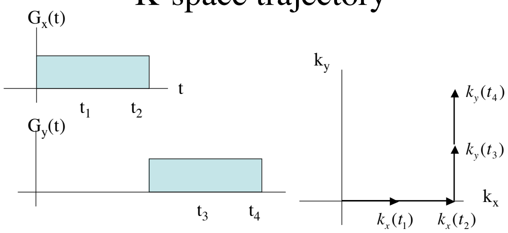


K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

K-space trajectory



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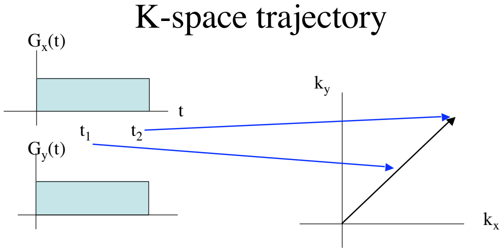
K-space trajectory



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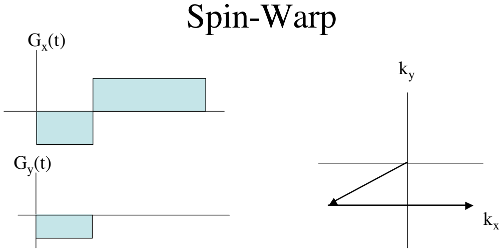
Spin-Warp



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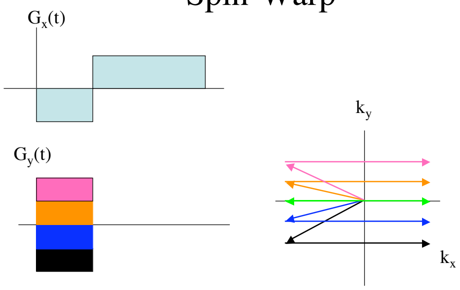

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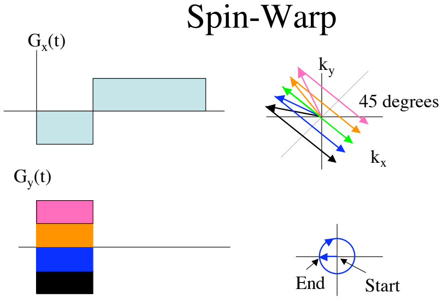
Spin-Warp



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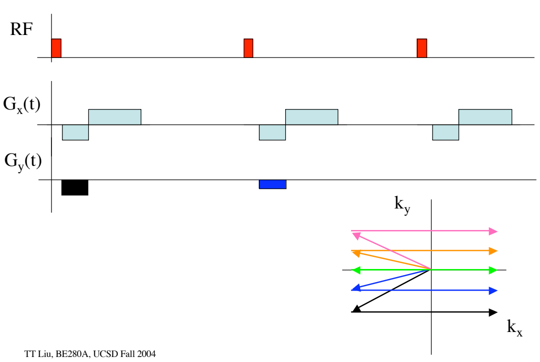
Spin-Warp Pulse Sequence



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