Bioengineering 280A
Principles of Biomedical Imaging

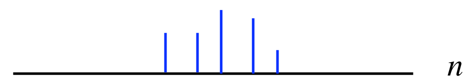
Fall Quarter 2004
Lecture 3
1D Fourier Transforms

Topics

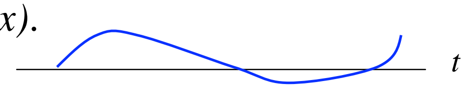
1. Signal Representations
2. Some Linear Algebra
3. 1D Fourier Transform
4. Transform Pairs
5. FT Properties

What is a signal?

Discrete-time/space signal: continuous valued function with a discrete time/space index, denoted as *s[n]*.

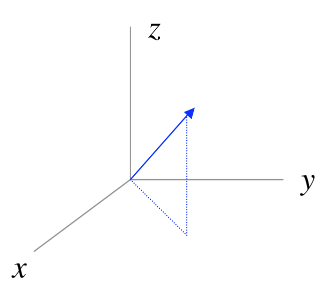


Continuous-time/space signal: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)*.



Signal Representation

It's easiest to start with discrete-time signals, which can be represented as vectors of either finite or infinite dimension. We'll start with finite dimensional vectors since they are easier to think about. Consider a finite-time signal with just 3 points. This can represented as a vector in $\Re^3$ for real-valued signals or C^3 for complex-valued signals.

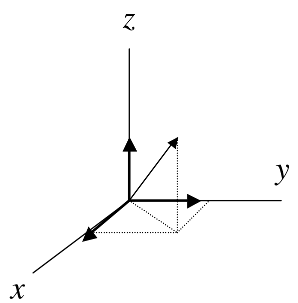


In signal notation: $s[n] = 1,1,1$

In vector notation: $\mathbf{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Basis Vectors

The numbers that we use to represent a signal depend on the choice of *basis vectors*, or more generally, *basis functions*.

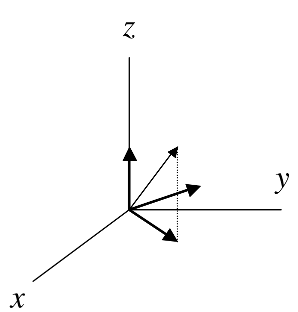


$$\mathbf{s} = \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}1 + \begin{bmatrix}0\\1\\0\end{bmatrix}1 + \begin{bmatrix}0\\0\\1\end{bmatrix}1$$

Here the unit vectors are used as the basis vectors. Note these are just Kronecker Delta functions!

Basis Vectors

Any 3 vectors that *span* 3-dimensional space may be used as basis vectors. Recall from linear algebra, that these 3 vectors must be linearly independent. In other words, any one basis vector cannot be expressed as a linear sum of the other basis vectors. For any basis set, the signal coefficients are simply the weights of the basis vectors.



$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2\\ \sqrt{2}/2\\ 0\end{bmatrix}\sqrt{2} + \begin{bmatrix}\sqrt{2}/2\\ -\sqrt{2}/2\\ 0\end{bmatrix}0 + \begin{bmatrix}0\\0\\1\end{bmatrix}1$$

With this set of basis vectors, the coefficients of the signal are $s[n] = \sqrt{2}, 0, 1$

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Inner Products

$$\langle r,s\rangle = \begin{cases} \sum_{n=1}^{N} r^*[n]s[n] & \text{for finite-dimensional vectors} \\ \sum_{n=-\infty}^{\infty} r^*[n]s[n] & \text{for infinite-dimensional vectors} \\ \int_{t=-\infty}^{\infty} r^*(t)s(t)dt & \text{for continuous signals} \end{cases}$$

The norm is defined as

$$\|s\| = \sqrt{\langle s,s\rangle}$$

Orthogonality

Some other notations for the inner product:

$$\langle \mathbf{x},\mathbf{y}\rangle = \mathbf{x} \bullet \mathbf{y} = \mathbf{x}^T\mathbf{y}$$

Also, recall that the angle between the two vectors is given by

$$\cos\theta = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

This gives rise to the famous Cauchy-Schwarz Inequality

$$|\langle \mathbf{x},\mathbf{y}\rangle| \le \|\mathbf{x}\|\|\mathbf{y}\|$$

Two vectors are orthogonal if $\langle \mathbf{x},\mathbf{y}\rangle = 0$, and therefore $\theta = \pi/2$.

Orthonormal basis

A set of vectors $S = \{\mathbf{b}_i\}$ forms an orthonormal basis, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for $i \neq j$, every basis vector is normalized to have unit length $\|\mathbf{b}_i\| = 1$, and any vector $\mathbf{y}$ in the space can be expressed as a linear combination of the basis vectors, i.e. $\mathbf{y} = \sum_k c_k \mathbf{b}_k$.

Finding Expansion Coefficients

Define the basis matrix as $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \end{bmatrix}$.

Then any vector $\mathbf{y} = \mathbf{Bc} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

Multiply both sides of the equation by $\mathbf{B}^{-1}$, to obtain $\mathbf{c} = \mathbf{B}^{-1}\mathbf{y}$.
Because the basis vectors are orthonormal $\mathbf{B}^T\mathbf{B} = \mathbf{I}$, and therefore $\mathbf{B}^{-1} = \mathbf{B}^T$. So, we can also write $\mathbf{c} = \mathbf{B}^T\mathbf{y}$.
By definition, $\mathbf{B}$ is an orthonormal or unitary matrix.

Expansion Coefficients

$$\mathbf{c} = \mathbf{B}^T\mathbf{y} = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_N^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{y} \rangle \\ \langle \mathbf{b}_2, \mathbf{y} \rangle \\ \vdots \\ \langle \mathbf{b}_N, \mathbf{y} \rangle \end{bmatrix}$$

For any vector **y,** the *i*th expansion coefficient is the inner product of the *i*th orthonormal basis vector with **y**.

Parseval's Theorem

$$\|\mathbf{c}\|^2 = \langle \mathbf{c}, \mathbf{c} \rangle = \mathbf{c}^T\mathbf{c} = \mathbf{y}^T\mathbf{B}\mathbf{B}^T\mathbf{y} = \mathbf{y}^T\mathbf{y} = \|\mathbf{y}\|^2$$

Exercise : Verify that $\mathbf{B}\mathbf{B}^T = \mathbf{I}$ for an orthonormal basis set. This is referred to as the resolution of unity or resolution of identity.

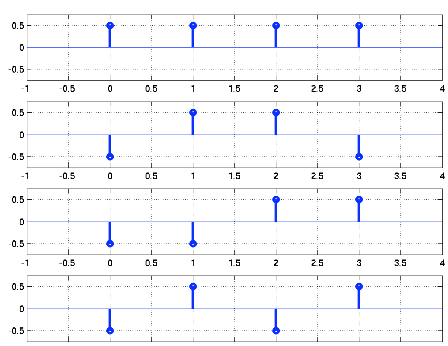
An orthonormal expansion preserves length.

Examples

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

Examples



$$\mathbf{B} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Is this an orthonormal set of basis functions?
What familiar set of functions do they correspond to?

Examples

$$y[n] = \frac{\sqrt{2}}{2}\cos(\pi(2n-3)/4) \text{ for } n = 0,1,2,3$$
$$= \quad -1/2, 1/2, 1/2, -1/2$$

$$\mathbf{c} = \mathbf{B}^T\mathbf{y} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Fourier Basis

$$x_m[\mathrm{n}] = \frac{1}{2}\exp(\text{-j}2\pi mn/4) \text{ for } n = 0,1,2,3$$

$$\mathbf{B} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

As an exercise, verify that $\mathbf{B}^{\mathrm{H}}\mathbf{B} = \mathbf{I}$, where H denotes a Hermitian tranpose -- i.e. conjugate every term and take the tranpose.

Recap - Finite Dimensional Case

Matrix Notation

$\mathbf{y} = \mathbf{Bc}$ where $\mathbf{c} = \mathbf{B}^T \mathbf{y}$

Signal notation

$$y[n] = \sum_{i=1}^{N} c_i b_i[n] = \sum_{i=1}^{N} \langle y[n], b_i[n] \rangle b_i[n]$$

Infinite Dimensional Expansions

Discrete-Time Series Expansion

$$y[n] = \sum_{i=-\infty}^{\infty} c_i b_i[n] \quad c_i = \langle b_i[n], y[n] \rangle$$

Continuous-Time Series Expansion

$$y(t) = \sum_{i=-\infty}^{\infty} c_i b_i(t) \quad c_i = \langle b_i(t), y(t) \rangle$$

Continuous-Time Integral Expansion

$$y(t) = \int_{-\infty}^{\infty} c_f b_f(t) df \quad c_f = \langle b_f(t), y(t) \rangle$$

Expansions with Delta Functions

Discrete-Time Series Expansion

$$y[n] = \sum_{k=-\infty}^{\infty} c_k \delta[k-n] \quad \text{where } c_k = \langle \delta[k-n], y[n] \rangle = y[k]$$
$$= \sum_{k=-\infty}^{\infty} y[k]\delta[k-n]$$

Continuous-Time Integral Expansion

$$y(t) = \int_{-\infty}^{\infty} c_\tau \delta(t-\tau)d\tau \quad \text{where } c_\tau = \int_{-\infty}^{\infty} y(\tau)\delta(t-\tau)d\tau = y(t)$$
$$= \int_{-\infty}^{\infty} y(\tau)\delta(t-\tau)d\tau$$

Imaging and Basis Functions

1. Most imaging methods may be considered to be the process of taking the inner product of an object with a set of basis functions, where the basis functions are determined by physics and engineering. In other words, the basis functions act as our "rulers" for measuring the object.
2. Fourier bases show up frequently because the world is full of harmonic oscillators, e.g. MRI.
3. The basis functions are not necessarily orthogonal.
4. In fact, the "basis" functions usually do not even form a complete basis, so that the best we can do is approximate the original object given our measurements.

Fourier Series Expansion

Basis functions are the complex exponentials

$$b_m(t) = \frac{1}{\sqrt{T}} e^{j2\pi m f_0 t} = \frac{1}{\sqrt{T}}\left(\cos 2\pi m f_0 t + j \sin 2\pi m f_0 t\right)$$

where f_0 is the fundamental frequency and $T_0 = 1/f_0$ is the fundamental period.

Are they orthonormal? Yes, over an interval defined by the period T_0.

$$\left\langle e^{j2\pi m f_0 t}, e^{j2\pi n f_0 t}\right\rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi(m-n)f_0 t}\,dt = \delta[m-n]$$

Continuous - time series expansion is:

$$g(t) = \sum_{m=-\infty}^{\infty} c_m b_m(t) = \frac{1}{\sqrt{T}}\sum_{m=-\infty}^{\infty} c_m e^{j2\pi m f_0 t}$$

The basis coefficients are:

$$c_m = \left\langle \frac{1}{\sqrt{T}} e^{j2\pi m f_0 t}, g(t)\right\rangle = \frac{1}{\sqrt{T}}\int_{-T_0/2}^{T_0/2} g(t) e^{-j2\pi m f_0 t}\,dt$$

Fourier Series Expansion

Note that we can write the Fourier Series Expansion in a more familiar form as…

$$\begin{aligned} g(t) &= \frac{1}{\sqrt{T}}\sum_{m=-\infty}^{\infty} c_m e^{j2\pi m f_0 t} \\ &= \frac{1}{\sqrt{T}}\sum_{m=-\infty}^{\infty} c_m (\cos 2\pi m f_0 t + j \sin 2\pi m f_0 t) \\ &= \frac{1}{\sqrt{T}}\left[c_0 + \sum_{m=1}^{\infty}\left(c_m + c_{-m}\right)\cos 2\pi m f_0 t + j\left(c_m - c_{-m}\right)\sin 2\pi m f_0 t\right] \\ &= \frac{1}{\sqrt{T}}\left[c_0 + \sum_{m=1}^{\infty} a_m \cos 2\pi m f_0 t + b_m \sin 2\pi m f_0 t\right] \end{aligned}$$

The Fourier Transform

Basis functions are complex exponentials $b_f(t) = e^{j2\pi ft}$

Are they orthonormal?

$$\left\langle e^{j2\pi f_1 t}, e^{j2\pi f_2 t} \right\rangle = \int_{-\infty}^{\infty} e^{j2\pi(f_2 - f_1)t} dt = \delta(f_2 - f_1)$$

Continuous - time integral expansion is :

$$g(t) = \int_{-\infty}^{\infty} G(f) b_f(t) df = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

The basis coefficients are :

$$G(f) = \left\langle e^{j2\pi ft}, g(t) \right\rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

The Fourier Transform

The Fourier Transform (FT) is simply given by the basis coefficients

$$G(f) = \left\langle e^{j2\pi ft}, g(t) \right\rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

The Inverse Fourier Transform is the continuous - time integral expansion for $g(t)$:

$$g(t) = \int_{-\infty}^{\infty} G(f) b_f(t) df = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

This can also be written as an inner product in Fourier Space

$$g(t) = \left\langle e^{-j2\pi ft}, G(f) \right\rangle$$

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Units

Temporal Coordinates, e.g. *t* in seconds, *f* in cycles/second

$$G(f) = \langle e^{j2\pi ft}, g(t) \rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \qquad \text{Fourier Transform}$$

$$g(t) = \langle e^{-j2\pi ft}, G(f) \rangle = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. *x* in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \langle e^{j2\pi k_x x}, g(x) \rangle = \int_{-\infty}^{\infty} g(x) e^{-j2\pi k_x x} dx \qquad \text{Fourier Transform}$$

$$g(x) = \langle e^{-j2\pi k_x x}, G(k_x) \rangle = \int_{-\infty}^{\infty} G(k_x) e^{j2\pi k_x x} dk_x \qquad \text{Inverse Fourier Transform}$$

Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F\left(\Pi\left(x\right)\right) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \sin c(k_x) \end{aligned}$$

Computing Transforms

$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

Computing Transforms

Similarly,

$$F\left\{e^{j2\pi k_0 x}\right\} = \delta(k_x - k_0)$$

$$F\left\{\cos 2\pi k_0 x\right\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$

$$F\left\{\sin 2\pi k_0 x\right\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$

Linearity

The Fourier Transform is linear.

$$F\{ag(x)+bh(x)\} = aG(k_x)+bH(k_x)$$

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Duality

Note the similarity between these two transforms

$$F\{e^{j2\pi ax}\} = \delta(k_x - a)$$
$$F\{\delta(x-a)\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\{G(x)\} = g(-k_x)$$

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Application of Duality

$$F\{\mathrm{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

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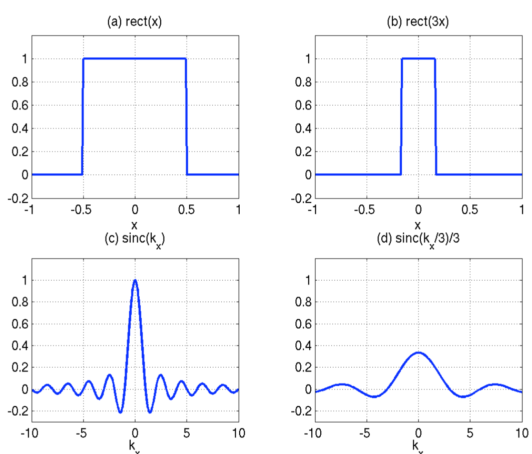
Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|} G\left(\frac{k_x}{a}\right)$$



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Shift Theorem

$$F\{g(x-a)\} = G(k_x)e^{-j2\pi a k_x}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.
Each frequency component is shifted by a. This corresponds to a relative phase shift of

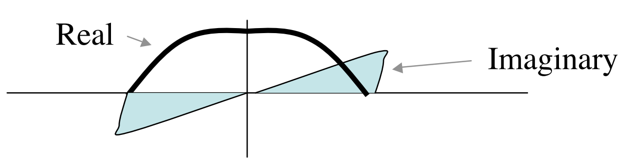
$$-2\pi a/(\text{spatial period}) = -2\pi a k_x$$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields $\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$

Hermitian Symmetry

$$\begin{aligned} F\{g^*(x)\} &= \int_{-\infty}^{\infty} g^*(x)e^{-j2\pi k_x x}dx \\ &= \left[\int_{-\infty}^{\infty} g(x)e^{j2\pi k_x x}dx\right]^* \\ &= G^*(-k_x) \end{aligned}$$

If $g(x)$ is real, then $g(x) = g^*(x)$, and therefore $G(k_x) = G^*(-k_x)$. $G(k_x)$ is said to exhibit Hermitian Symmetry. The real part of $G(k_x)$ is symmetric, while the imaginary part is anti-symmetric.



Convolution/Modulation Theorem

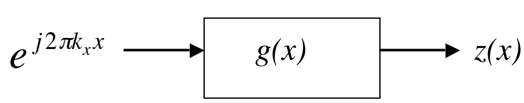
$$
\begin{aligned}
F\{g(x)*h(x)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u)*h(x-u)du\right]e^{-j2\pi k_x x}dx \\
&= \int_{-\infty}^{\infty} g(u)\int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu \\
&= \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du \\
&= G(k_x)H(k_x)
\end{aligned}
$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x)*H(k_x)$$

Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

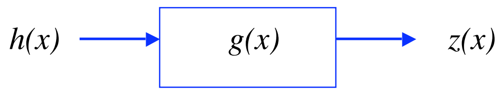


$$
\begin{aligned}
z(x) &= g(x)*e^{j2\pi k_x x} \\
&= \int_{-\infty}^{\infty} g(u)e^{j2\pi k_x (x-u)}du \\
&= G(k_x)e^{j2\pi k_x x}
\end{aligned}
$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

Eigenfunctions

Now consider an arbitrary input *h(x)*.



Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

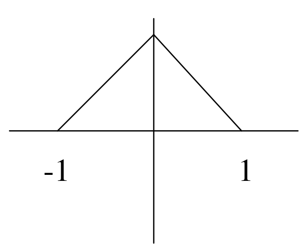
Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & otherwise \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1} (1 - |x|) e^{-j2\pi k_x x} dx = ??$$



Application of Convolution Thm.

Modulation

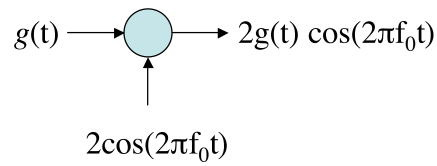
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G\left(k_x - k_0\right)$$

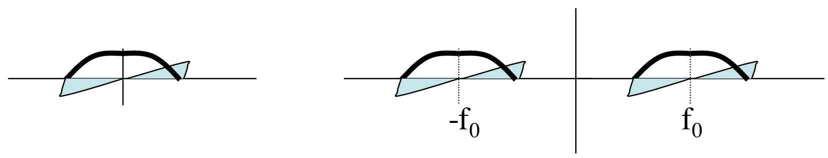
$$F\left[g(x)\cos\left(2\pi k_0 x\right)\right] = \frac{1}{2}G\left(k_x - k_0\right) + \frac{1}{2}G\left(k_x + k_0\right)$$

$$F\left[g(x)\sin\left(2\pi k_0 x\right)\right] = \frac{1}{2j}G\left(k_x - k_0\right) - \frac{1}{2j}G\left(k_x + k_0\right)$$

Example

Amplitude Modulation (e.g. AM Radio)





G(f)   G(f-f$_0$)+ G(f+f$_0$)

Parseval's Theorem

Recall that an orthonormal expansion preserves length or equivalently energy.

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(k_x)|^2 dk_x$$

The more general form of this theorem is

$$\int_{-\infty}^{\infty} g(x)h^*(x)dx = \int_{-\infty}^{\infty} G(k_x)H^*(k_x)dk_x$$

Parseval's Theorem Derivation

From the modulation theorem and the fact that $F\{h^*(x)\} = H^*(-k_x)$

we can write $F\{g(x)h^*(x)\} = G(k_x) * H^*(-k_x)$

$$\int_{-\infty}^{\infty} g(x)h^*(x)e^{-j2\pi k_x x}dx = \int_{-\infty}^{\infty} G(k_x - u)H^*(-u)du$$

Set $k_x = 0$ to obtain

$$\int_{-\infty}^{\infty} g(x)h^*(x)dx = \int_{-\infty}^{\infty} G(-u)H^*(-u)du$$

which yields the general form of the Parseval's formula

$$\int_{-\infty}^{\infty} g(x)h^*(x)dx = \int_{-\infty}^{\infty} G(k_x)H^*(k_x)dk_x$$

Setting $h(x) = g(x)$ then yields the more familiar form

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(k_x)|^2 dk_x$$