Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
Lecture 4
2D Fourier Transforms

Topics

1. 2D Signal Representations
2. 2D Fourier Transform
3. Transform Pairs
4. FT Properties

2D Signal

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Image Decomposition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} g[m,n] &= a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1] \\ &= \sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l] \\ &= \sum_{k=0}^{1}\sum_{l=0}^{1} c_{k,l} b_{k,l}[m,n] \end{aligned}$$

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Orthonormal Basis Functions

Discrete

$$\begin{aligned} \left\langle b_{k,l}, b_{k',l'} \right\rangle &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b_{k,l}^{*}[m,n] b_{k',l'}[m,n] \\ &= \delta[k-k', l-l'] \end{aligned}$$

Continuous

$$\begin{aligned} \left\langle b_{k_x,k_y}, b_{k_x',k_y'} \right\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_{k_x,k_y}^{*}(x,y) b_{k_x',k_y'}(x,y)\,dx\,dy \\ &= \delta(k_x-k_x', k_y-k_y') \end{aligned}$$

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Example

Are these orthonormal?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Example

Are these orthonormal?

1/2	1/2
1/2	1/2

1/2	-1/2
1/2	-1/2

1/2	1/2
-1/2	-1/2

1/2	-1/2
-1/2	1/2

Discrete Expansion Coefficients

The discrete expansion is

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{k,l} b_{k,l}[m,n]$$

If the basis functions $b_{k,l}[m,n]$ are orthonormal then

$$\begin{aligned}
c_{k,l} = \langle b_{k,l}, g\rangle &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b^*_{k,l}[m,n]g[m,n] \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b^*_{k,l}[m,n] \sum_{k'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} c_{k',l'} b_{k',l'}[m,n] \\
&= \sum_{k'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} c_{k',l'} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b^*_{k,l}[m,n] b_{k',l'}[m,n] \\
&= \sum_{k'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} c_{k',l'}\delta[k-k', l-l'] \\
&= c_{k,l}
\end{aligned}$$

Continuous Expansion Coefficients

The continuous expansion is

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(k_x,k_y) b_{k_x,k_y}(x,y)\,dk_x dk_y$$

If the basis functions $b_{k_x,k_y}(x,y)$ are orthonormal then

$$\begin{aligned}
c(k_x,k_y) = \langle b_{k_x,k_y}, g\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^*_{k_x,k_y}(x,y) g(x,y)\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^*_{k_x,k_y}(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(k'_x,k'_y) b_{k'_x,k'_y}(x,y)\,dk'_x dk'_y dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(k'_x,k'_y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^*_{k_x,k_y}(x,y) b_{k'_x,k'_y}(x,y)\,dxdy dk'_x dk'_y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(k'_x,k'_y)\delta(k_x-k'_x, k_y-k'_y)\,dk'_x dk'_y \\
&= c(k_x,k_y)
\end{aligned}$$

Separable Basis Functions

Discrete

$b_{k,l}[m,n] = b_k[m]b_l[n]$

$e.g. \quad \delta[m-k, n-l] = \delta[m-k]\delta[n-l]$

Continuous

$b_{k_x,k_y}(x,y) = b_{k_x}(x)b_{k_y}(y)$

$e.g. \quad \delta(x - x_i, y - y_i) = \delta(x - x_i)\delta(y - y_i)$

Separable Basis Functions

$b_1[n] = [1 \quad 0]$

$b_1[m] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

1	0
0	0

$b_2[n] = [0 \quad 1]$

$b_1[m] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

0	1
0	0

$b_1[n] = [1 \quad 0]$

$b_2[m] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

0	0
1	0

$b_2[n] = [0 \quad 1]$

$b_2[m] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

0	0
0	1

Separable Basis Functions

$b_1[n] = [1 \quad 1]/\sqrt{2}$

$b_1[m] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1/2	1/2
1/2	1/2

$b_2[n] = [1 \quad -1]/\sqrt{2}$

$b_1[m] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1/2	-1/2
1/2	-1/2

$b_1[n] = [1 \quad 1]/\sqrt{2}$

$b_1[m] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1/2	1/2
-1/2	-1/2

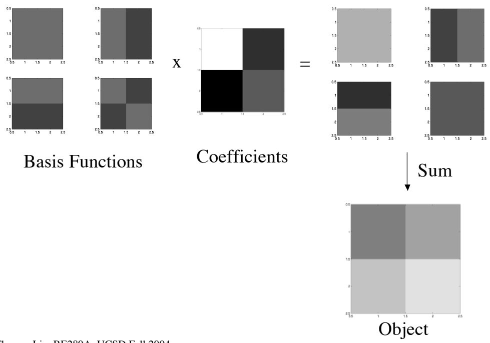
$b_2[n] = [1 \quad -1]/\sqrt{2}$

$b_1[m] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1/2	-1/2
-1/2	1/2

$b_{k,l}[m,n] = b_k[m]\ b_l[n] \qquad b_k[m] = \exp(-\pi mk)/\sqrt{2} \qquad b_l[n] = \exp(-\pi nl)/\sqrt{2}$

Example



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Fourier Basis Functions

Recall that for 1D the basis functions are complex exponentials

$$b_{k_x}(x) = e^{j2\pi k_x x}$$

For 2D, we use the separable 2D functions

$$b_{k_x,k_y}(x,y) = b_{k_x}(x)b_{k_y}(y) = e^{j2\pi k_x x}e^{j2\pi k_y y} = e^{j2\pi(k_x x + k_y y)}$$

Are they orthonormal?

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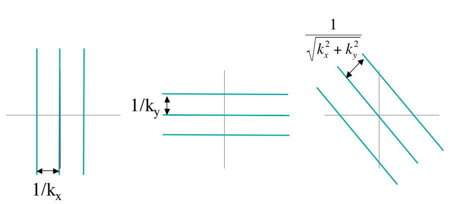
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

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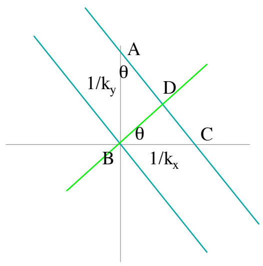
Plane Waves



$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2} + \frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \left\langle e^{j2\pi(k_x x + k_y y)}, g\right\rangle = \int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$

Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$
The Fourier Transform is then separable as well.

$$\begin{aligned} G(k_x,k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy \\ &= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx \int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy \\ &= G_X(k_x)G_Y(k_y) \end{aligned}$$

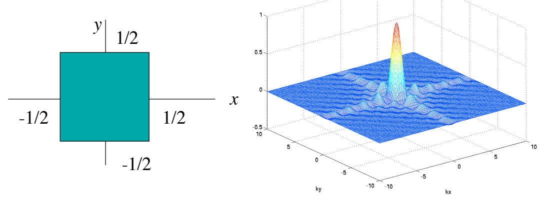
Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Example (sinc/rect)

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$



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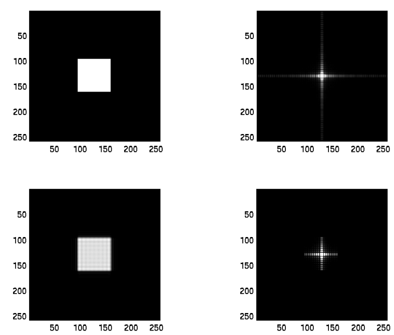
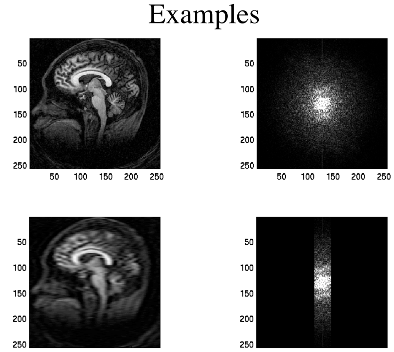
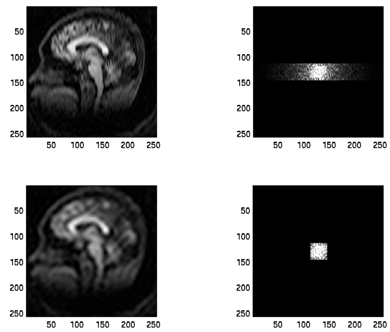
Example (sinc/rect)



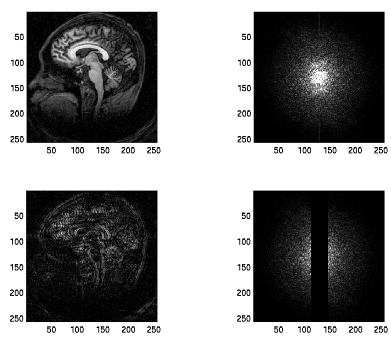
Examples



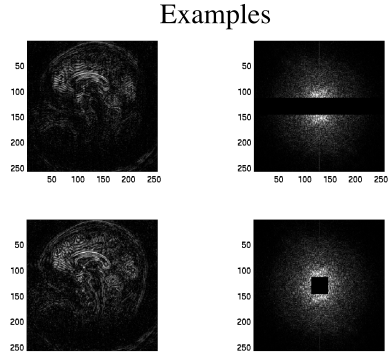
Examples



Examples



Examples



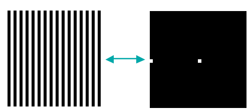
Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

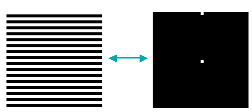
$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

Examples

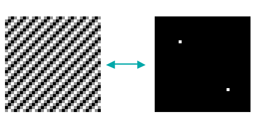
$$g(x,y) = 1 + e^{-j2\pi ax}$$
$$G(k_x,k_y) = \delta\left(k_x,k_y\right) + \delta(k_x + a)\delta(k_y)$$



$$g(x,y) = 1 + e^{j2\pi ay}$$
$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x)\delta(k_y - a)$$

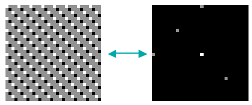


Examples



$$g(x,y) = \cos(2\pi(ax + by))$$
$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

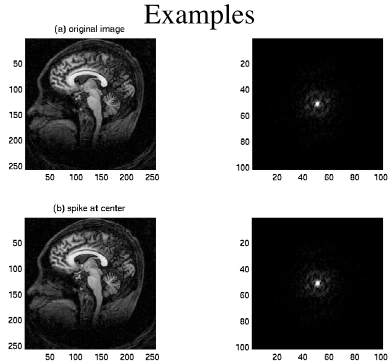
Examples



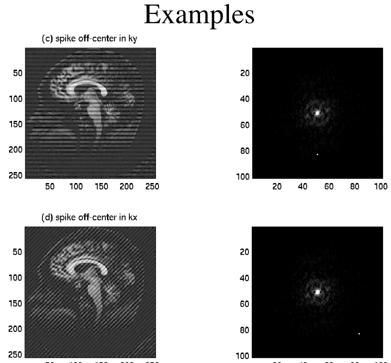
$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x + c)\delta(k_y) + \delta(k_x)\delta(k_y - d) + \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

$g(x,y) = ???$

Examples



Examples



Basic Properties

Linearity

$$F\left[ag(x,y)+bh(x,y)\right]=aG(k_x,k_y)+bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right]=\frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

Shift

$$F\left[g(x-a,y-b)\right]=G(k_x,k_y)e^{-j2\pi(k_x a+k_y b)}$$

Modulation

$$F\left[g(x,y)e^{j2\pi(xa+yb)}\right]=G(k_x-a,k_y-b)$$

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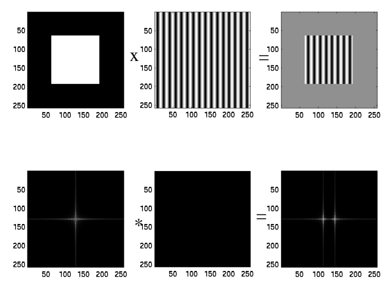
Modulation Example



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Convolution/Multiplication

Convolution

$$F\left[g(x,y)**h(x,y)\right]=G(k_x,k_y)H(k_x,k_y)$$

Multiplication

$$F\left[g(x,y)h(x,y)\right]=G(k_x,k_y)**H(k_x,k_y)$$

Multiplication in one domain translates into convolution in the other domain.

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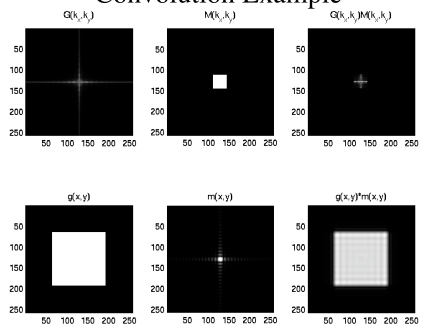
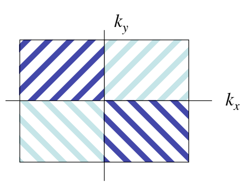
Convolution Example



Symmetry

$F\left[g^*(x,y)\right] = G^*(-k_x,-k_y)$

If $g(x,y)$ is real then $g(x,y) = g^*(x,y)$, so
$G^*(-k_x,-k_y) = G(k_x,k_y)$



Parseval's Relations

Energy is preserved

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(x,y)|^2\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(k_x,k_y)|^2\,dk_x dk_y$$

So is the inner product

$\langle g,h\rangle = \langle G,H\rangle$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g^*(x,y)h(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G^*(k_x,k_y)H(k_x,k_y)dk_x dk_y$$