Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
Lecture 5
Sampling

Topics

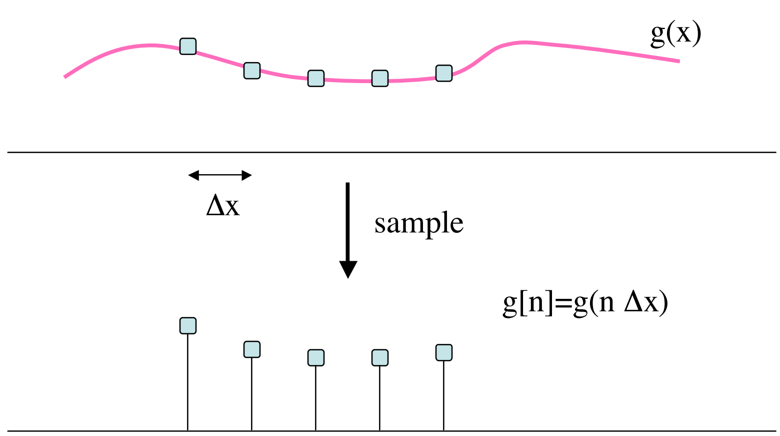
1. Overview of Sampling
2. 1D Sampling
3. 2D Sampling
4. Aliasing

Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

The Process of Sampling



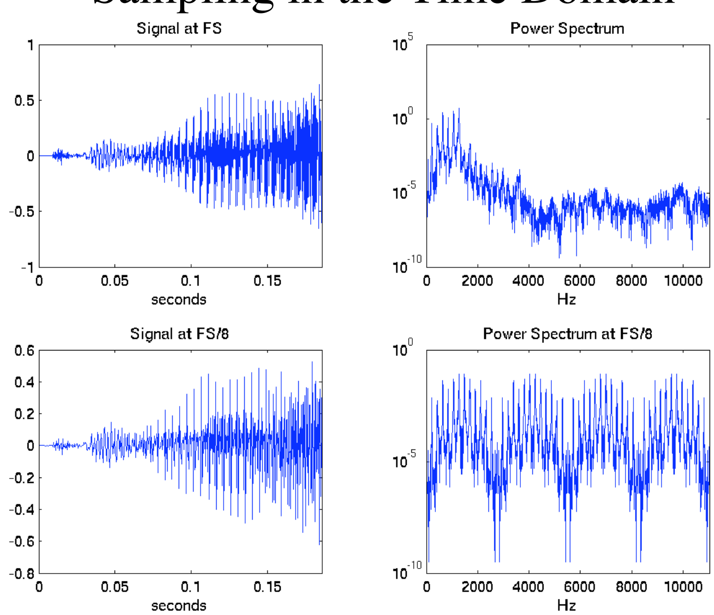
Questions

How finely do we need to sample?

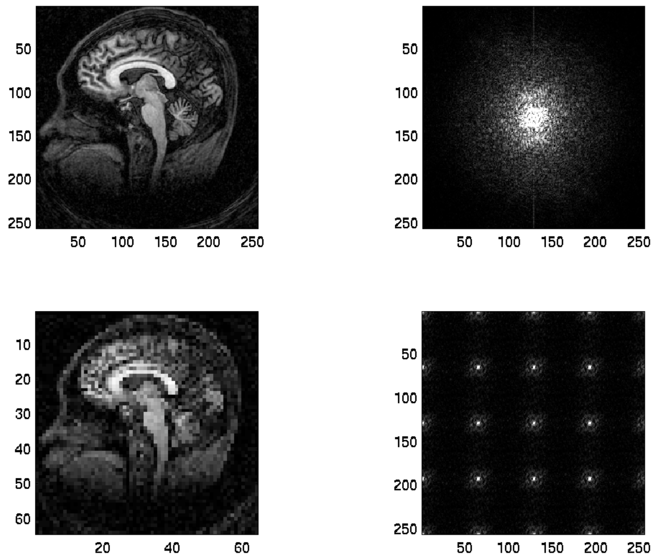
What happens if we don’t sample finely enough?

Can we reconstruct the original signal or image from its samples?

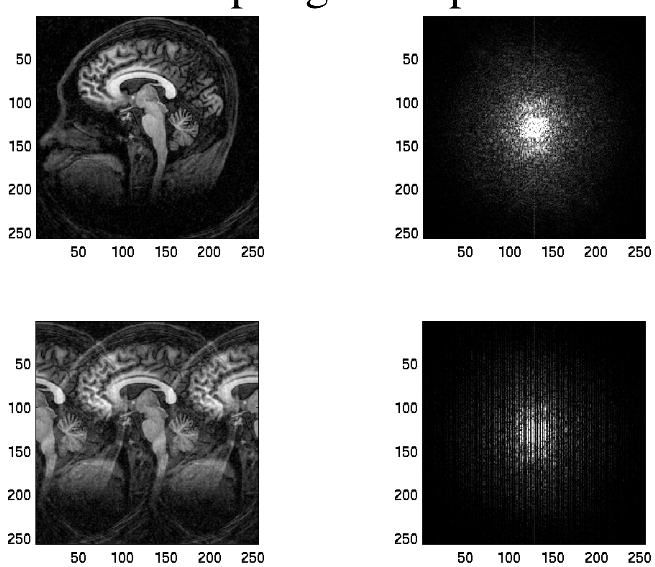
Sampling in the Time Domain



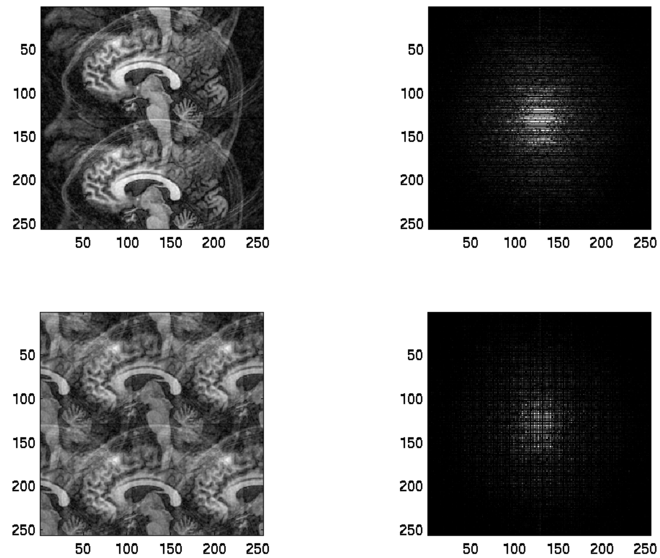
Sampling in Image Space



Sampling in k-space

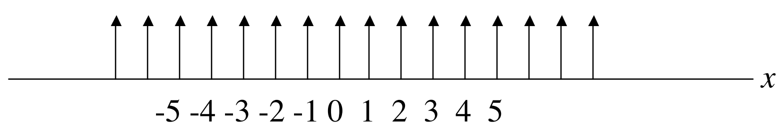


Sampling in k-space



Comb Function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



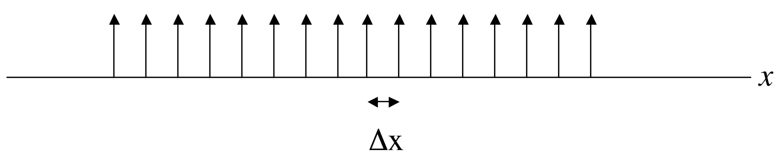
Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$

$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty} \delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

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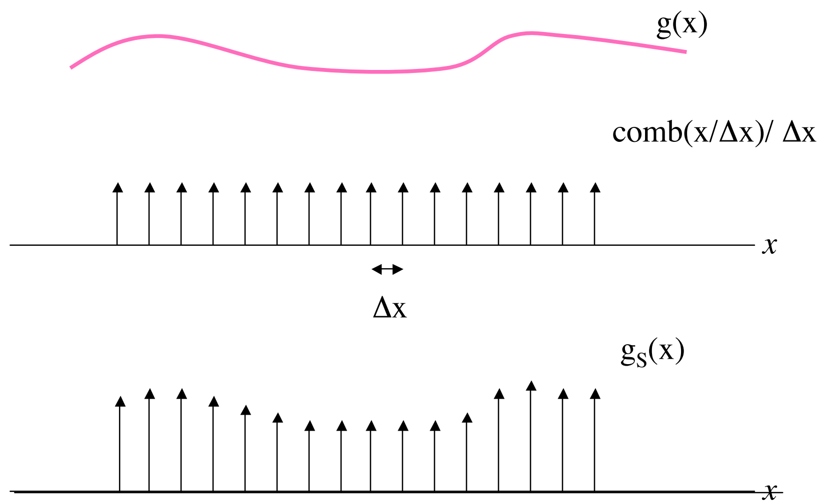
1D spatial sampling

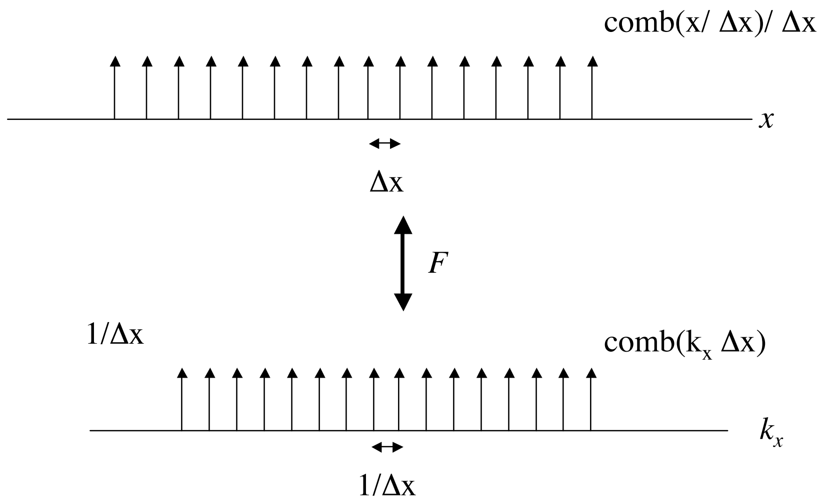


Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x} \Delta x comb(k_x \Delta x)$$
$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$
$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x})$$

Fourier Transform of comb(x/ Δx)



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Fourier Transform of $g_S(x)$

$$
\begin{aligned}
F\left[g_S(x)\right] &= F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G\left(k_x - \frac{n}{\Delta x}\right)
\end{aligned}
$$

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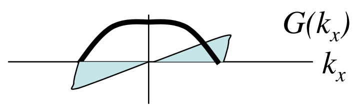
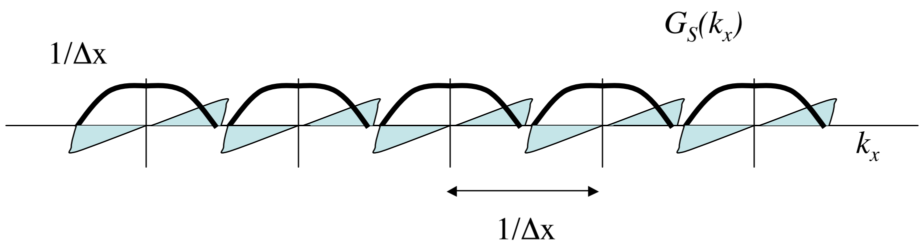
Fourier Transform of $g_S(x)$





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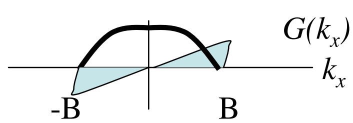
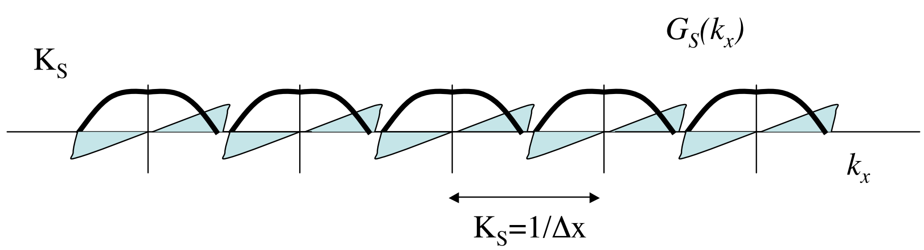
Nyquist Condition





To avoid overlap, we require that $1/\Delta x > 2B$ or
$K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

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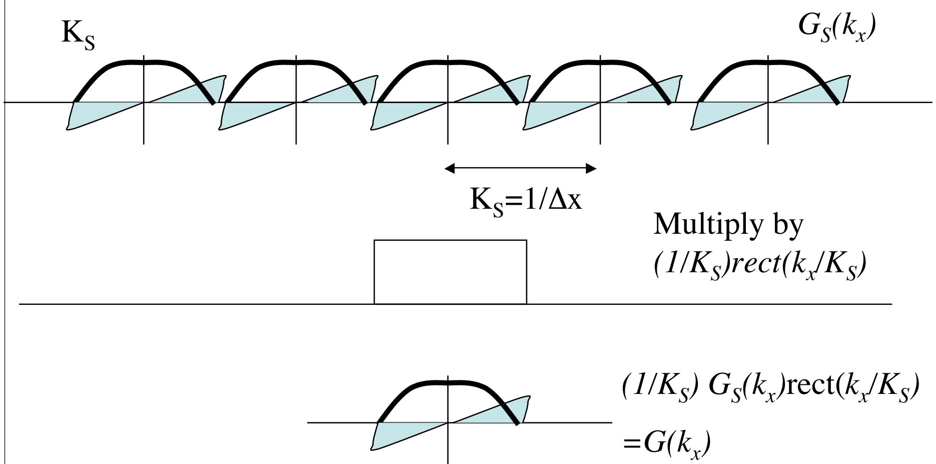
Reconstruction from Samples



Reconstruction from Samples

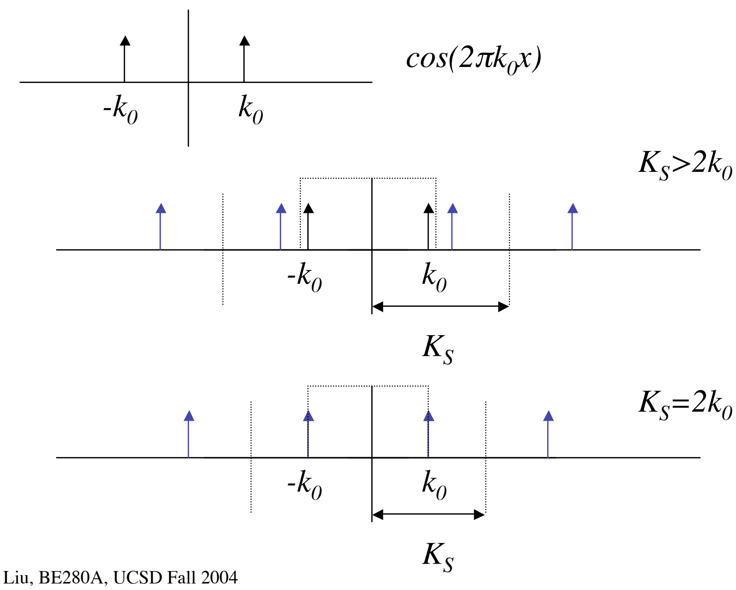
If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x / K_S) = G(k_x)$$

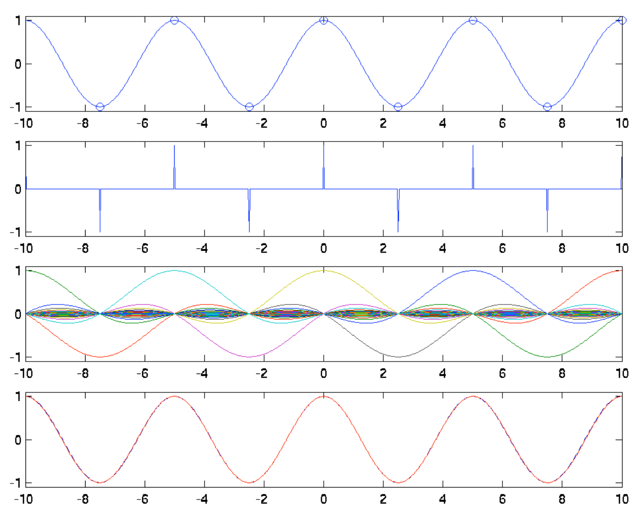
And the signal can be reconstructed by convolving the sample with a sinc function

$$\begin{aligned} \hat{g}_S(x) &= g_S(x) * \mathrm{sin}c(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X) * \mathrm{sin}c(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\,\mathrm{sin}c(K_s(x - n\Delta x)) \end{aligned}$$

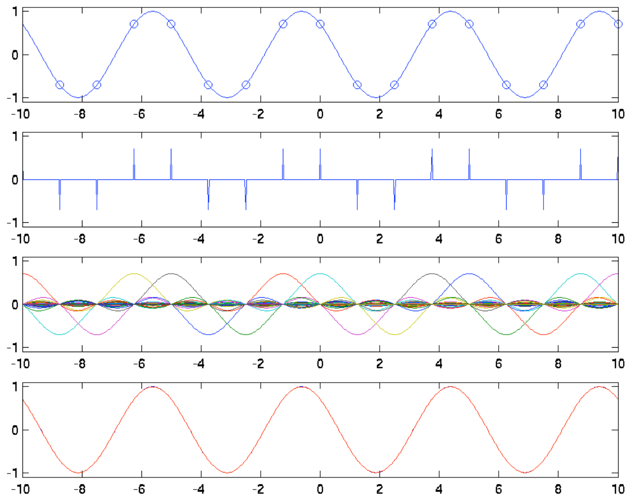
Example Cosine Reconstruction



Cosine Example with $K_S=2k_0$



Example with $K_s=4k_0$



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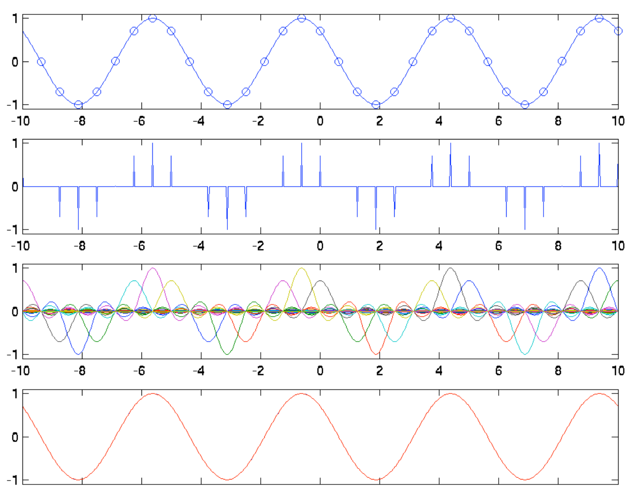
Example with $K_s=8k_0$



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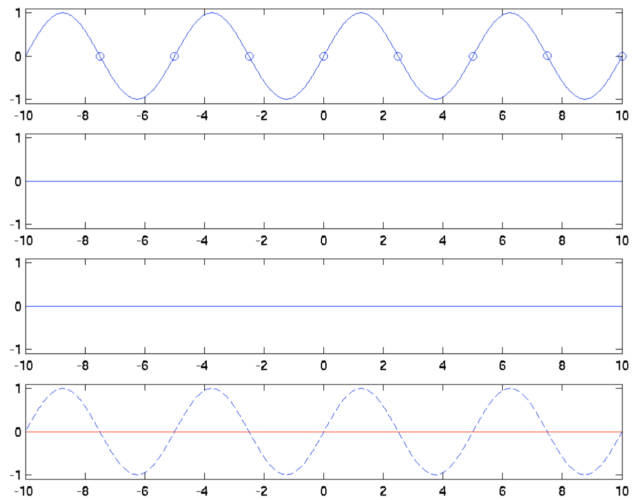
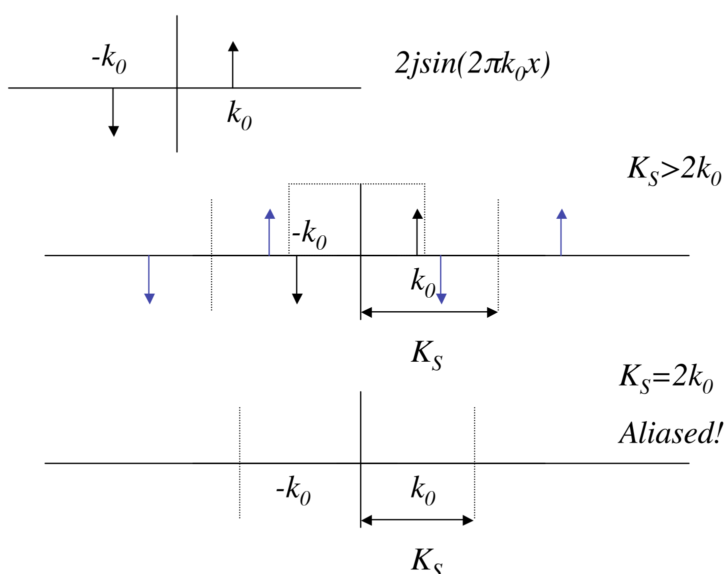
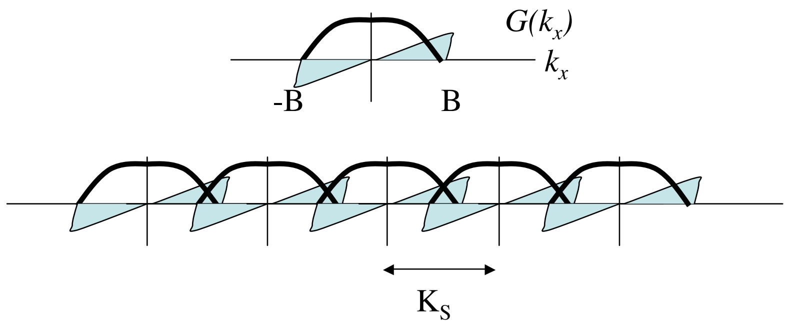
Sine Example with $K_s=2k_0$



Example Sine Reconstruction



Aliasing



Aliasing occurs when the Nyquist condition is not satisfied.
This occurs for $K_S \leq 2B$

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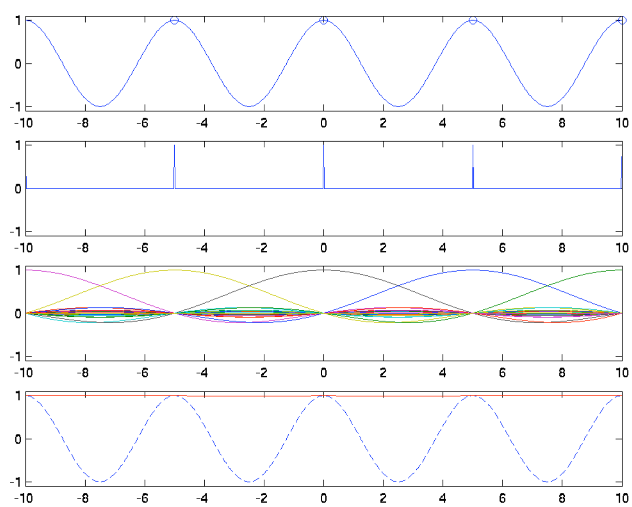
Aliasing Example



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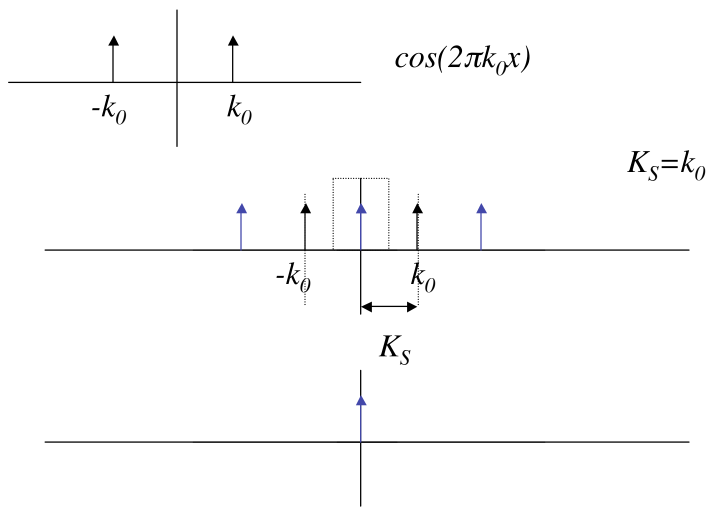
Aliasing Example



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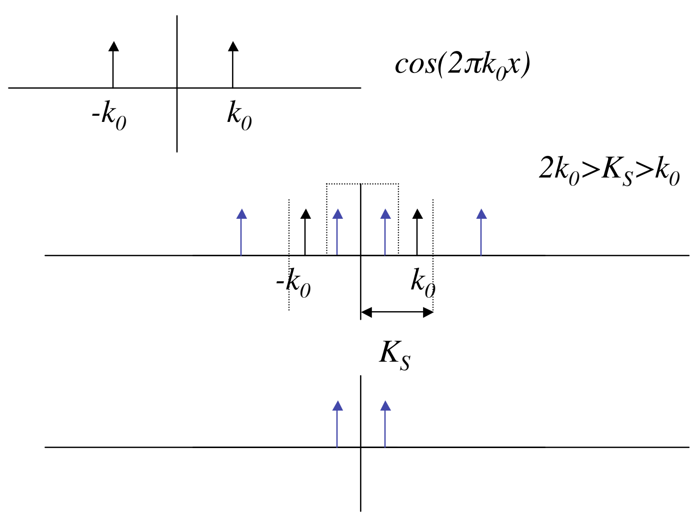
Aliasing Example



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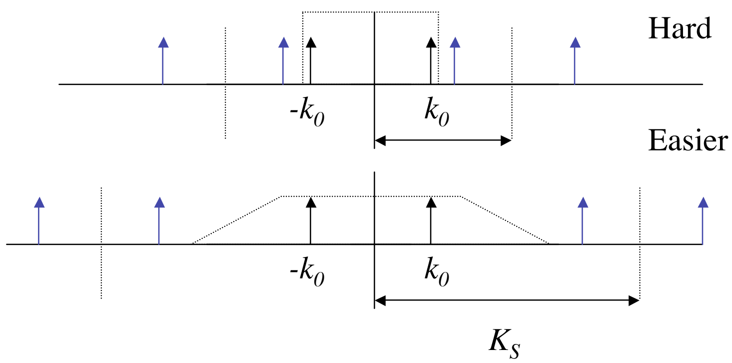
Practical Considerations

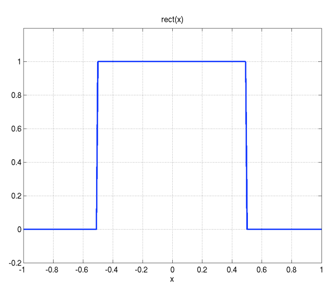
Why sample higher than the Nyquist frequency?

-- true sinc interpolation is not practical since the sinc function goes from -infinity to infinity

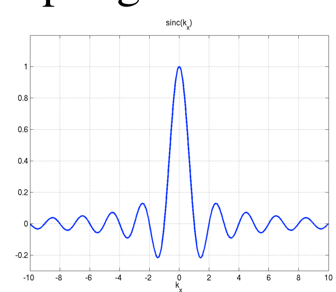
-- the requirements on the low-pass filter are reduced.



Fourier Sampling



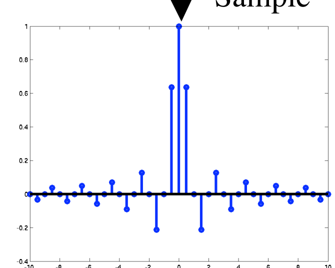
F



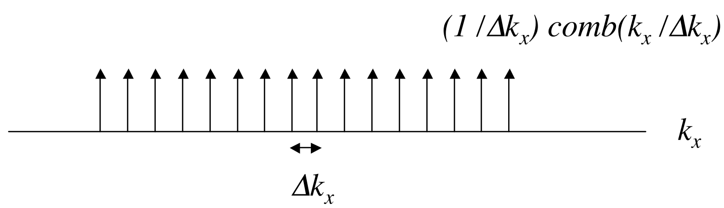
Sample

Instead of sampling the signal, we sample its Fourier Transform

??? F^{-1}



Fourier Sampling

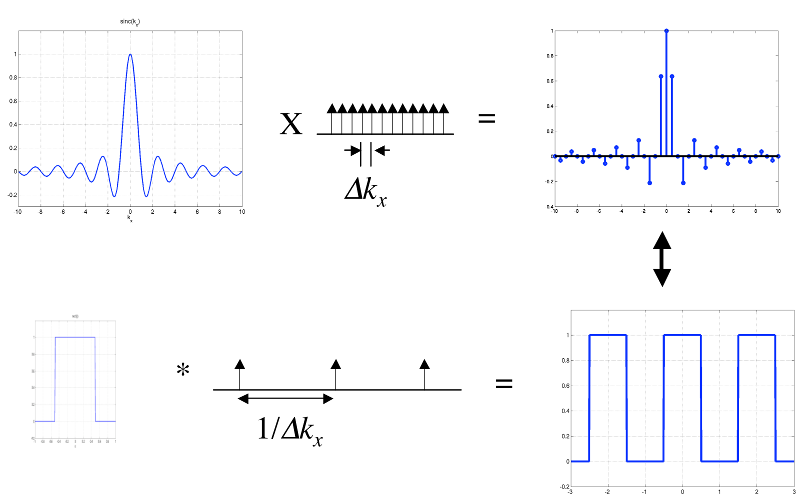


$$G_S(k_x) = G(k_x)\frac{1}{\Delta k_x}comb\left(\frac{k_x}{\Delta k_x}\right)$$

$$= G(k_x)\sum_{n=-\infty}^{\infty}\delta(k_x - n\Delta k_x)$$

$$= \sum_{n=-\infty}^{\infty}G(n\Delta k_x)\delta(k_x - n\Delta k_x)$$

Fourier Sampling -- Inverse Transform



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Fourier Sampling -- Inverse Transform

$$
\begin{aligned}
g_S(x) &= F^{-1}\left[G_S(k_x)\right] \\
&= F^{-1}\left[G(k_x)\frac{1}{\Delta k_x}comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= F^{-1}\left[G(k_x)\right] * F^{-1}\left[\frac{1}{\Delta k_x}comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= g(x) * comb\left(x\Delta k_x\right) \\
&= g(x) * \sum_{n=-\infty}^{\infty}\delta(x\Delta k_x - n) \\
&= g(x) * \frac{1}{\Delta k_x}\sum_{n=-\infty}^{\infty}\delta(x - \frac{n}{\Delta k_x}) \\
&= \frac{1}{\Delta k_x}\sum_{n=-\infty}^{\infty}g(x - \frac{n}{\Delta k_x})
\end{aligned}
$$

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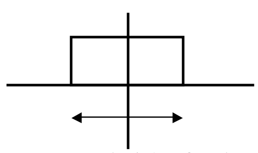
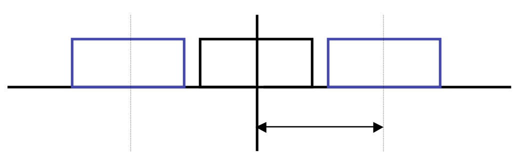
Nyquist Condition



FOV (Field of View)



$1/\Delta k_x$

To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/$FOV

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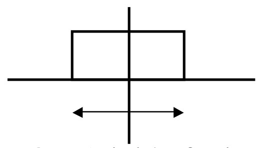
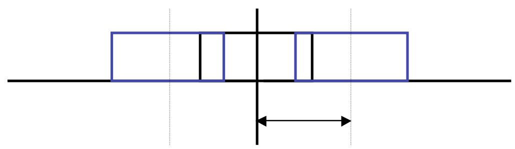
Aliasing



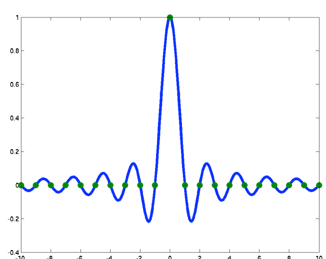
FOV (Field of View)



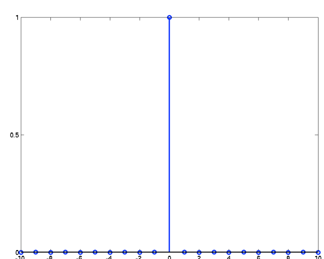
$1/\Delta k_x$

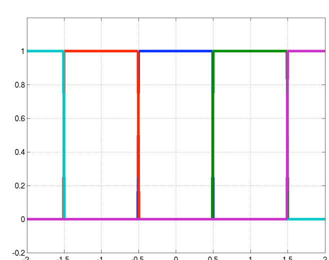
Aliasing occurs when $1/\Delta k_x < FOV$

Aliasing Example



$\Delta k_x = 1$

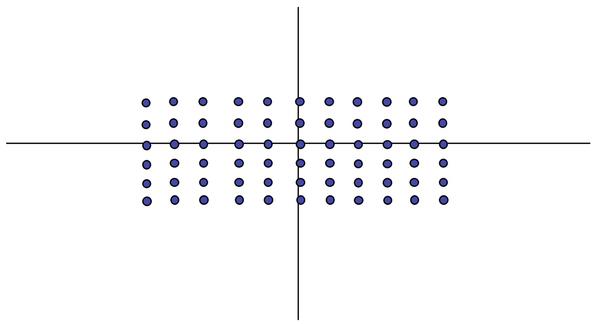




$1/\Delta k_x = 1$

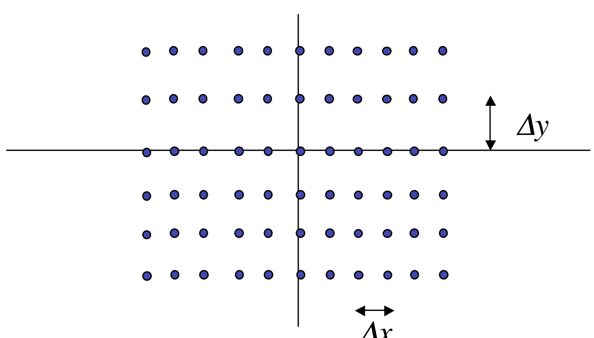
2D Comb Function

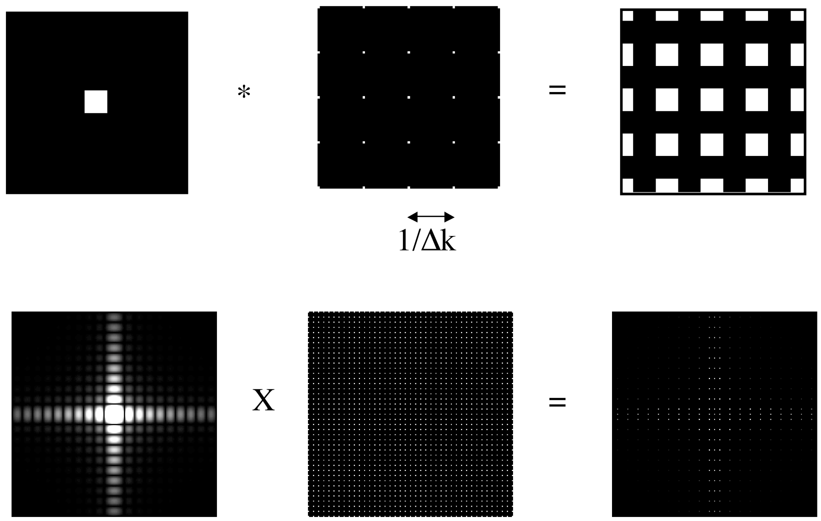
$$
\begin{aligned}
comb(x,y) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n) \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m)\delta(y-n) \\
&= comb(x)comb(y)
\end{aligned}
$$



Scaled 2D Comb Function

$$
\begin{aligned}
comb(x/\Delta x, y/\Delta y) &= comb(x/\Delta x)comb(y/\Delta y) \\
&= \Delta x \Delta y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x)\delta(y-n\Delta y)
\end{aligned}
$$





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2D k-space sampling

$$
\begin{aligned}
G_S(k_x,k_y) &= G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) \\
&= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)
\end{aligned}
$$

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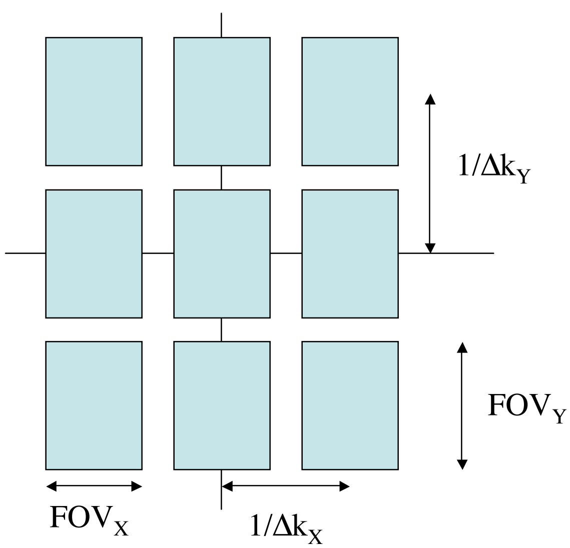
2D k-space sampling

$$
\begin{aligned}
g_S(x,y) &= F^{-1}\left[G_S(k_x,k_y)\right] \\
&= F^{-1}\left[G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right] \\
&= F^{-1}\left[G(k_x,k_y)\right] * F^{-1}\left[\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right] \\
&= g(x,y) ** comb\left(x\Delta k_x\right)comb\left(y\Delta k_y\right) \\
&= g(x) ** \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x\Delta k_x - m)\delta(y\Delta k_y - n) \\
&= g(x) ** \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - \frac{m}{\Delta k_x})\delta(y - \frac{n}{\Delta k_y}) \\
&= \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(x - \frac{m}{\Delta k_x}, y - \frac{n}{\Delta k_y})
\end{aligned}
$$

Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

Aliasing