Bioengineering 280A
Principles of Biomedical Imaging

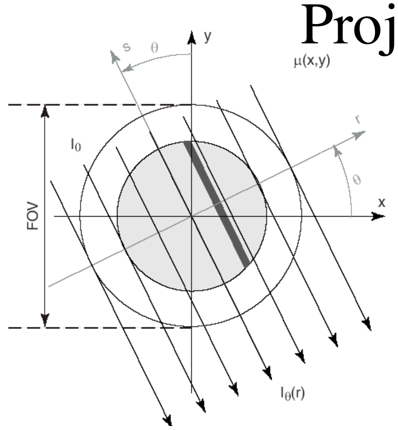
Fall Quarter 2004
X-Rays/CT Lecture 2



# Topics

- Review
- Filtered Backprojection
- Fan Beam
- Spiral CT
- Applications

# Projections




$$I_\theta(r) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(x,y)\,ds\right)$$

$$= I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)\,ds\right)$$



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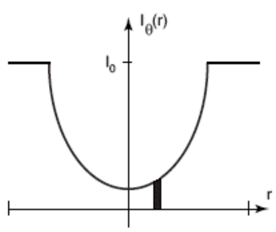
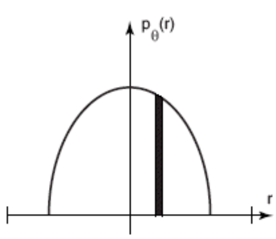
# Projections




(b)

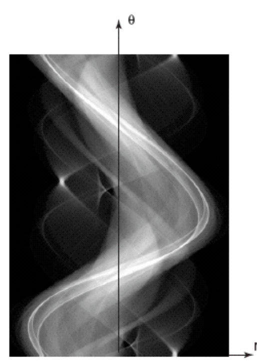



(c)

$$I_\theta(r) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)\,ds\right)$$

$$p_\theta(r) = -\ln \frac{I_\theta(r)}{I_0}$$

$$= \int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)\,ds$$




Sinogram



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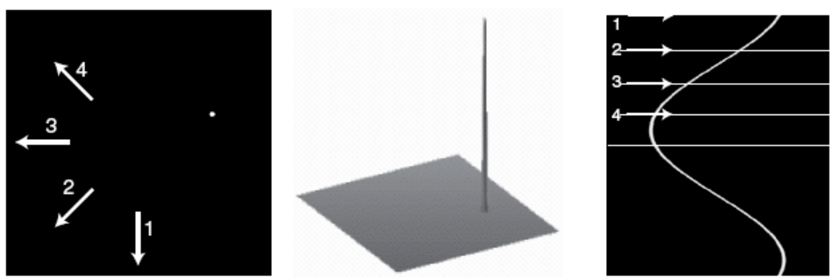
# Sinogram




(a) (b) (c)

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# Direct Inverse Approach

| | | |
|---|---|---|
| $\mu_1$ | $\mu_2$ | $p_1$ |
| $\mu_3$ | $\mu_4$ | $p_2$ |
| $p_3$ | $p_4$ | |

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_4 = \mu_2 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns.
Are these the correct equations to use?

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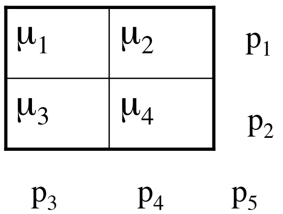
# Direct Inverse Approach



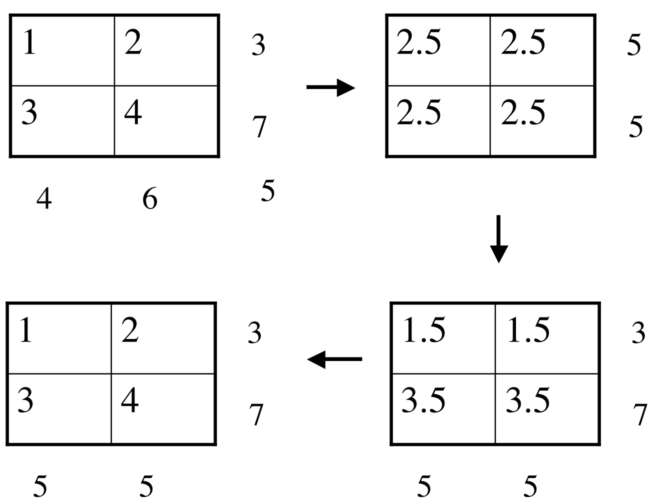

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_5 = \mu_1 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

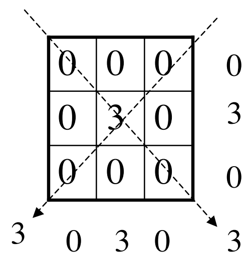
4 equations, 4 unknowns. These are linearly independent now.
In general for a NxN image, $N^2$ unknowns, $N^2$ equations.
This requires the inversion of a $N^2 x N^2$ matrix
For a high-resolution 512x512 image, $N^2$=262144 equations.
Requires inversion of a 262144x262144 matrix!
Inversion process sensitive to measurement errors.



# Iterative Inverse Approach
## Algebraic Reconstruction Technique (ART)



# Backprojection




| | | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 0 | 0 | 0 |

→

| | | |
|---|---|---|
| 1 | 0 | 0 |
| 1 | 2 | 1 |
| 0 | 0 | 1 |

→

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 3 | 1 |
| 0 | 1 | 1 |

→

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 4 | 1 |
| 1 | 1 | 1 |



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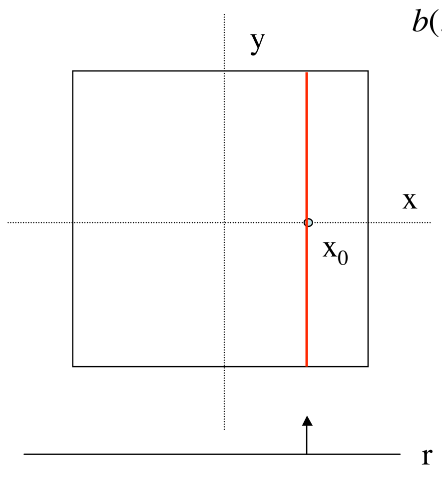
# Backprojection




$$b(x,y) = B\{p(r,\theta)\}$$
$$= \int_0^{\pi} p(x\cos\theta + y\sin\theta,\theta)\,d\theta$$

$$b(x_0,y) = p(r,\theta=0)\Delta\theta$$
$$= p(x_0)\Delta\theta$$



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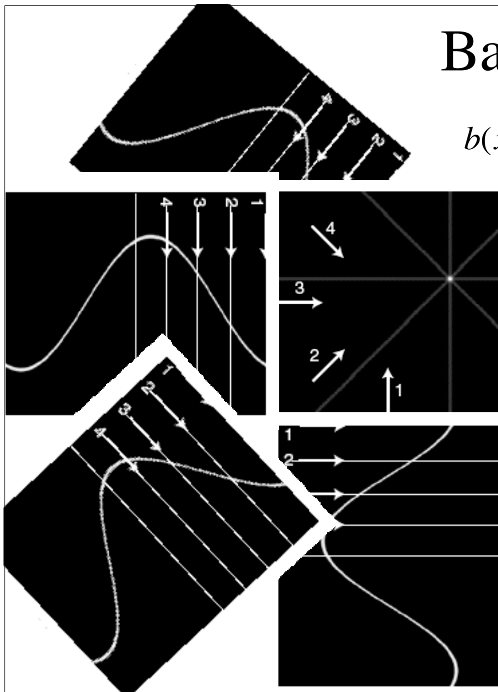
# Backprojection



$$b(x,y) = B\{p(r,\theta)\}$$
$$= \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$



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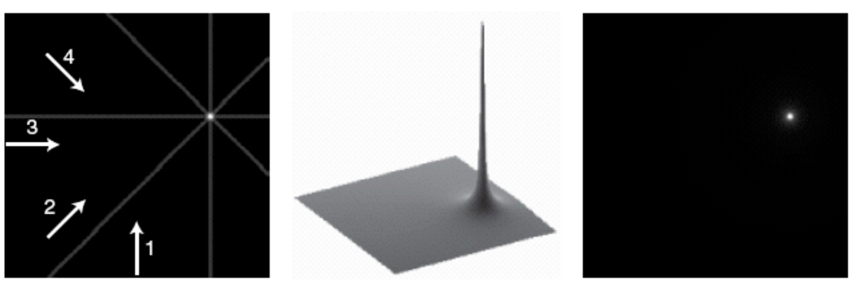
# Backprojection




$$b(x,y) = B\{p(r,\theta)\} = \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$



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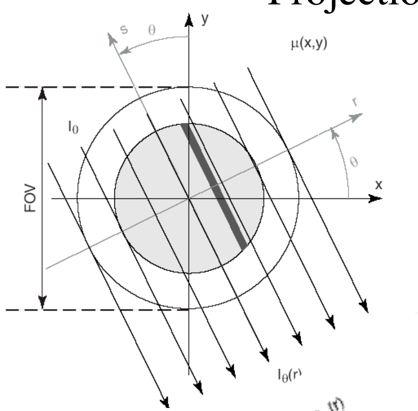
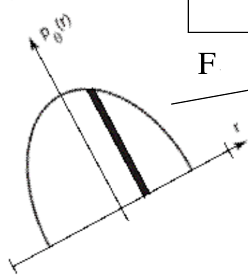
# Projection Theorem




$$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= F_{2D}\left[\mu(x,y)\right]$$

$$\boxed{\begin{aligned} U(k_x,k_y) &= P(k,\theta) \\ k_x &= k\cos\theta \\ k_y &= k\sin\theta \\ k &= \sqrt{k_x^2 + k_y^2} \end{aligned}}$$




$$\xrightarrow{F} \quad P(k,\theta) = \int_{-\infty}^{\infty} p_\theta(r)e^{-j2\pi kr}dr$$



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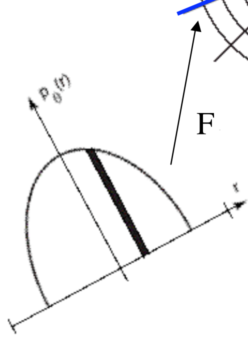
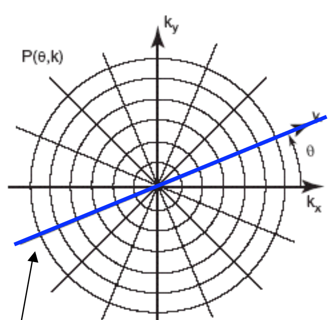
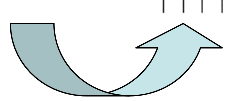
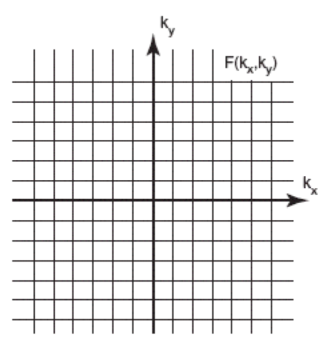
# Fourier Reconstruction






Interpolate onto Cartesian grid
then take inverse transform



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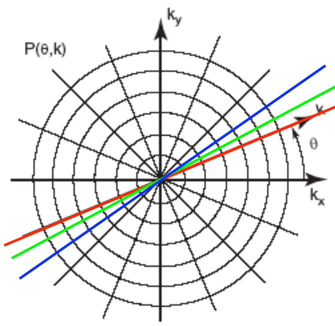
## Fourier Interpretation



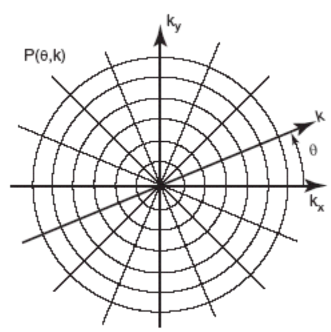

$$\text{Density} \approx \frac{\text{N}}{\text{circumference}} \approx \frac{\text{N}}{2\pi|\text{k}|}$$

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.



Kak and Slaney; Suetens 2002

## Polar Version of Inverse FT




$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(k_x,k_y)e^{j2\pi(k_x x+k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} U(k,\theta)e^{j2\pi(k\cos\theta x \;+k\sin\theta y)}kdkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} U(k,\theta)e^{j2\pi(xk\cos\theta \;+yk\sin\theta)}|k|dkd\theta$$



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# Filtered Backprojection

$$\mu(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} U(k,\theta) e^{j2\pi(xk\cos\theta \; + yk\sin\theta)} |k| dk d\theta$$

$$= \int_0^{\pi} \int_{-\infty}^{\infty} |k| U(k,\theta) e^{j2\pi kr} dk d\theta$$

$$= \int_0^{\pi} u^*(r,\theta) d\theta \longleftarrow \text{Backproject a filtered projection}$$

$$where\ r = x\cos\theta + y\sin\theta$$

$$u^*(r,\theta) = \int_{-\infty}^{\infty} |k| U(k,\theta) e^{j2\pi kr} dk$$

$$= u(r,\theta) * F^{-1}\left[|k|\right]$$

$$= u(r,\theta) * q(r)$$



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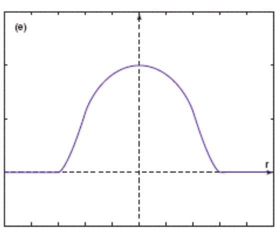
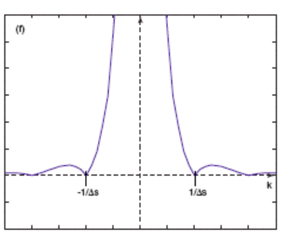
# Reconstruction Path

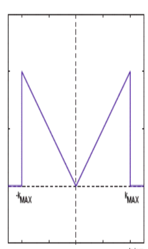
Projection




F ↓




x



$\xrightarrow{F^{-1}}$ Filtered Projection $\longrightarrow$ Back-Project



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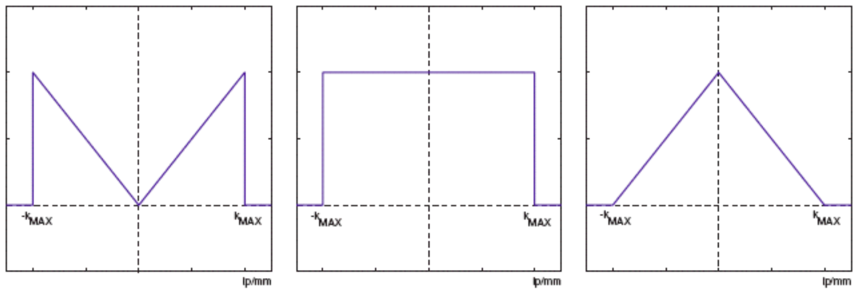
# Ram-Lak Filter




$$q(r) = F^{-1}\left[|k|\right] = \int_{-\infty}^{\infty} |k| e^{j2\pi kr}\, dk$$

Not a realistic convolution kernel.

Ram-Lak Filter

$$q(r) = F^{-1}\left[|k|\, rect\left(\frac{k}{2k_{\max}}\right)\right] = \int_{-k_{\max}}^{k_{\max}} |k| e^{j2\pi kr}\, dk$$

$k_{max}=1/\Delta s$

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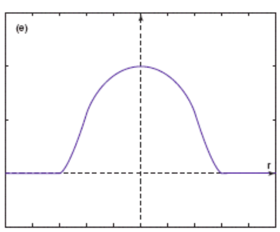
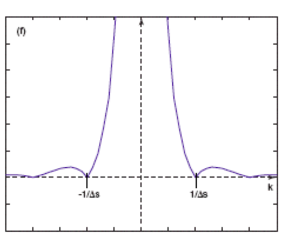
# Reconstruction Path

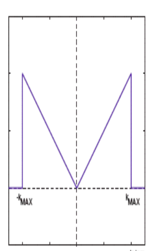
Projection




F ↓




x



$\xrightarrow{F^{-1}}$ Filtered Projection ⟶ Back-Project

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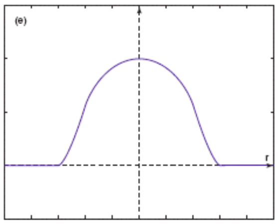
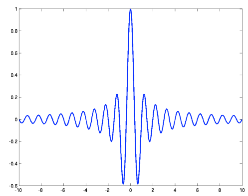
# Reconstruction Path

Projection



*



→ Filtered Projection

↓

Back-Project



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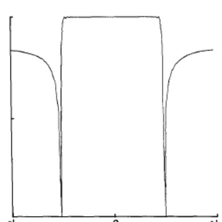
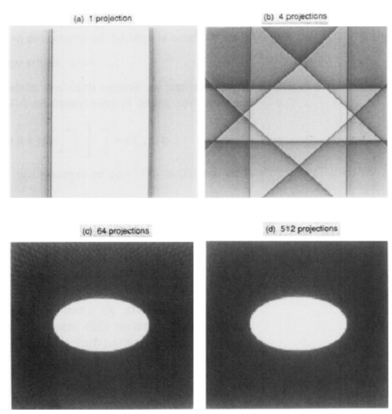
# Example





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# Additional Filtering

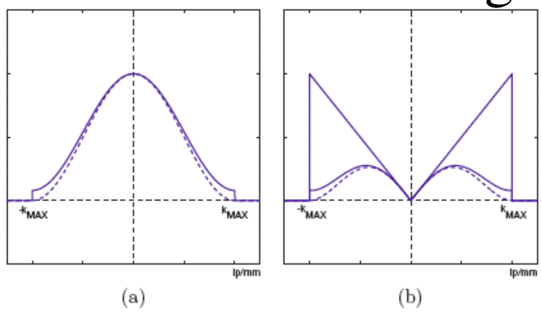
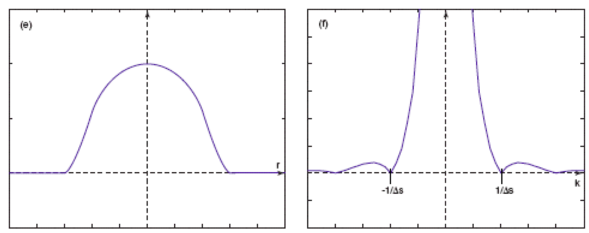



Figure 5.12: *(a) Hamming window with $\alpha = 0.54$ and Hanning window (dashed) with $\alpha = 0.5$. (b) Ramp filter and its products with a Hamming window and a Hanning window (dashed).*

$k_{max}=1/\Delta s$

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# Sampling Requirements

How many detectors do we need?

How many angular views do we need?

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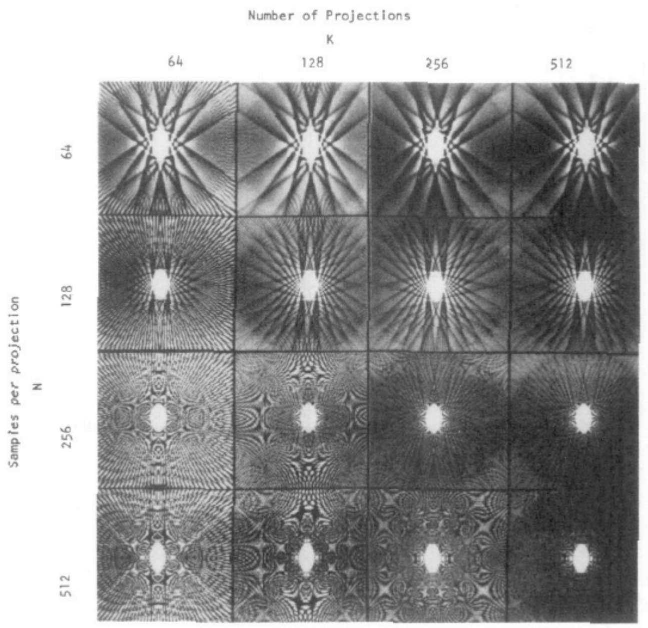


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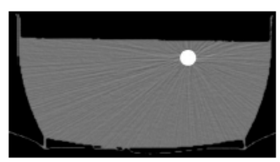
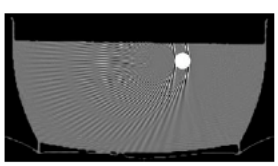
# Artifacts

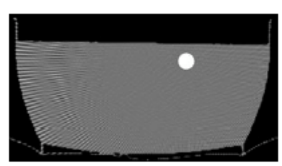
Object



Effect of Noise



Aliasing due to insufficient number of detectors



Aliasing due to insufficient number of views

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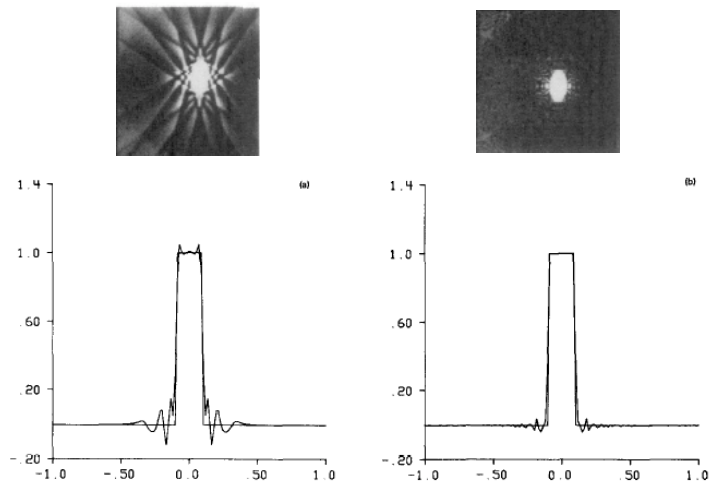
# Projection Aliasing



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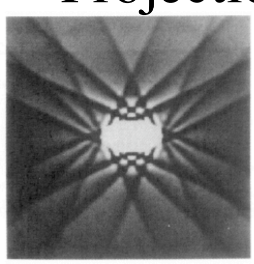
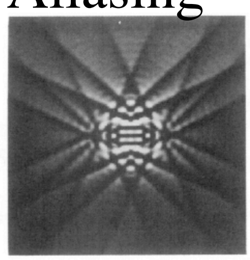
# Projection Aliasing

Aliased Image

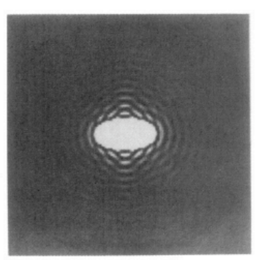


(a)



(b)

Alias Component



(aliased image)-
(alias component)



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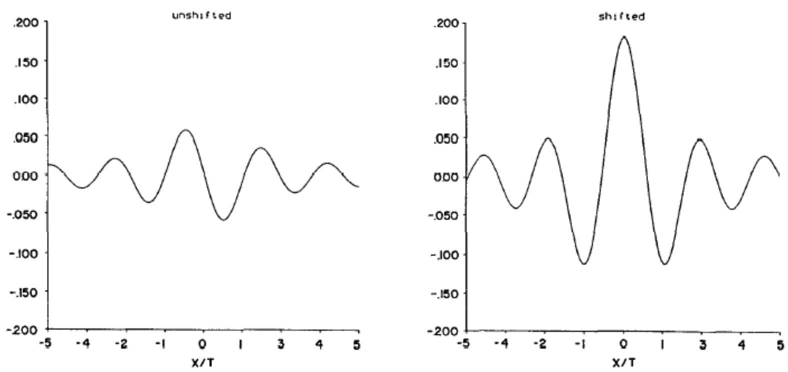
# Alias components



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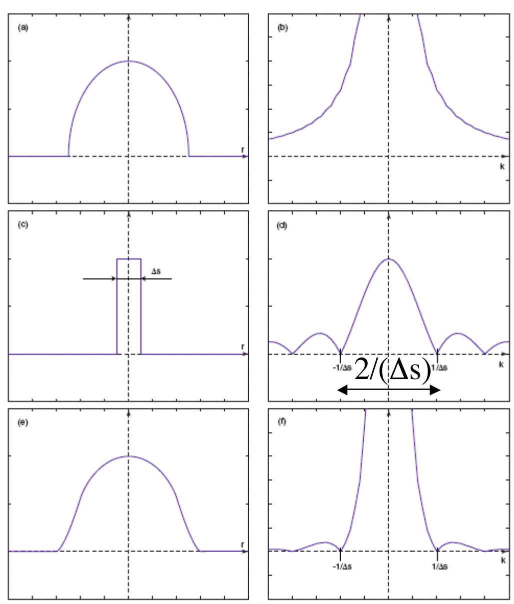
# Sampling Requirements

Projection

Beam Width

Smoothed Projection




W= 2/(Δs)
δ=1/W= Δs/2



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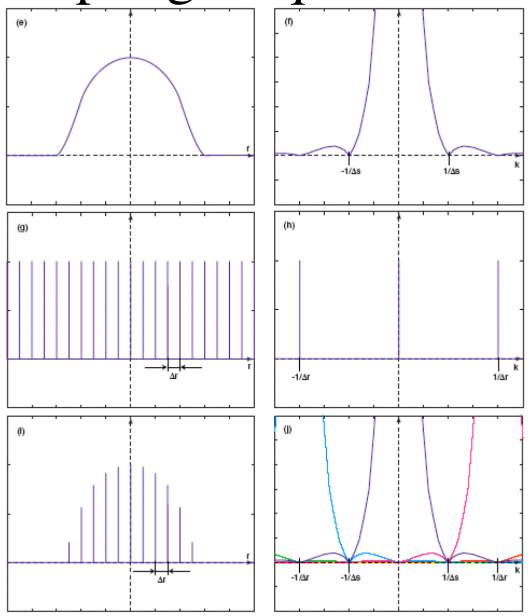
# Sampling Requirements

Smoothed Projection

Detectors
$\Delta r \leq \Delta s/2$

Sampled Smooth Projection



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# Detector Sampling Requirements

Beamwidth of detector $\Delta s$

Sampling interval $\Delta r$

Requirement is $\Delta r \leq \Delta s/2$



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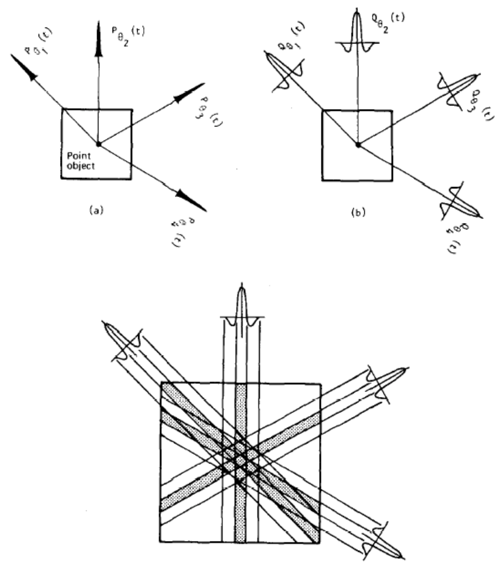
# View Aliasing

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# View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

$$\frac{\pi FOV}{N_{views}} = \Delta r$$

$$N_{views,360} = \frac{\pi FOV}{\Delta r} = \pi N_{proj} \quad \text{(for 360 degrees)}$$

$$N_{views,180} = \frac{\pi N_{proj}}{2} \quad \text{(for 180 degrees)}$$

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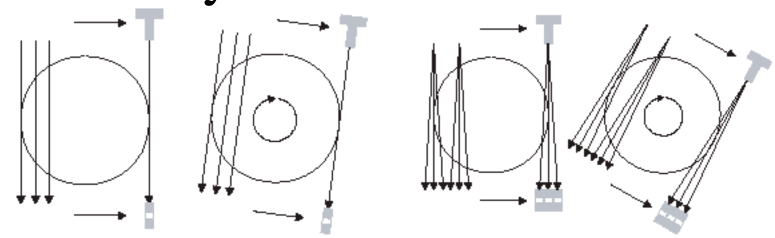
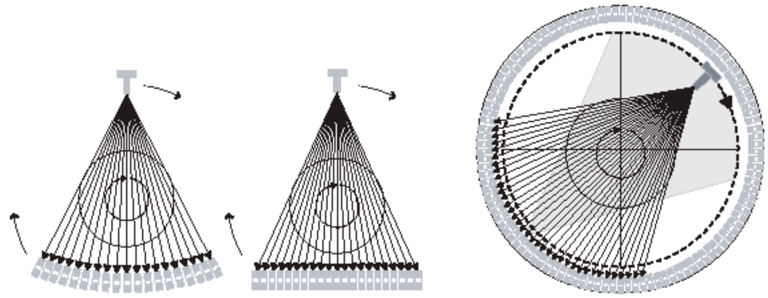
# CT System Generations



(a) (b)

5 minutes/slice

20 seconds /slice



(c) (d)

0.5 seconds /slice



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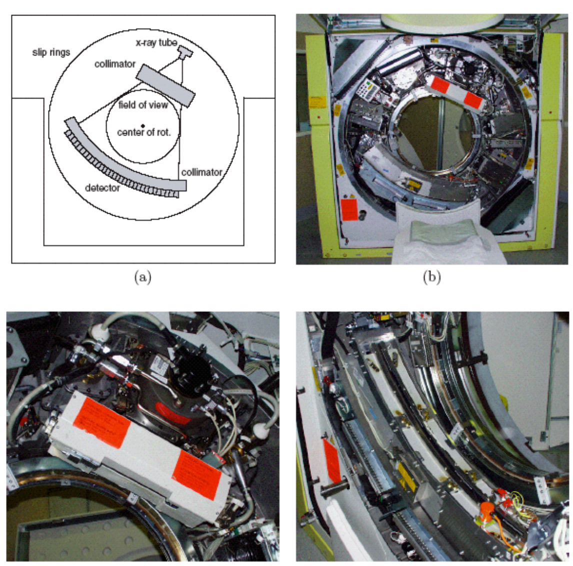
# CT System




(a) (b) (c) (d)



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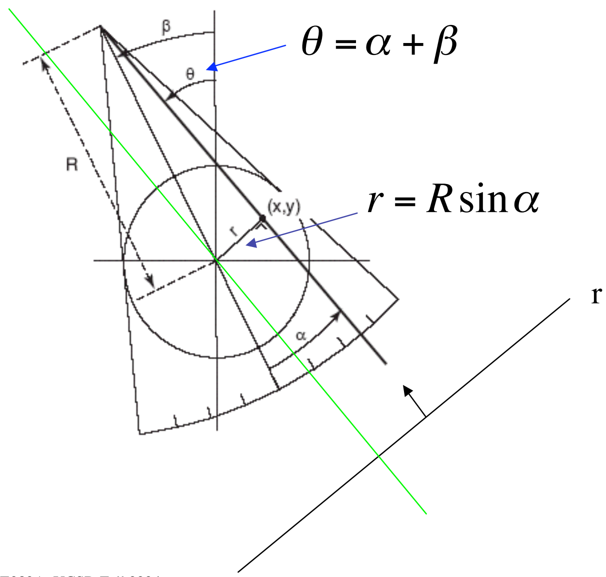
# Fan Beam

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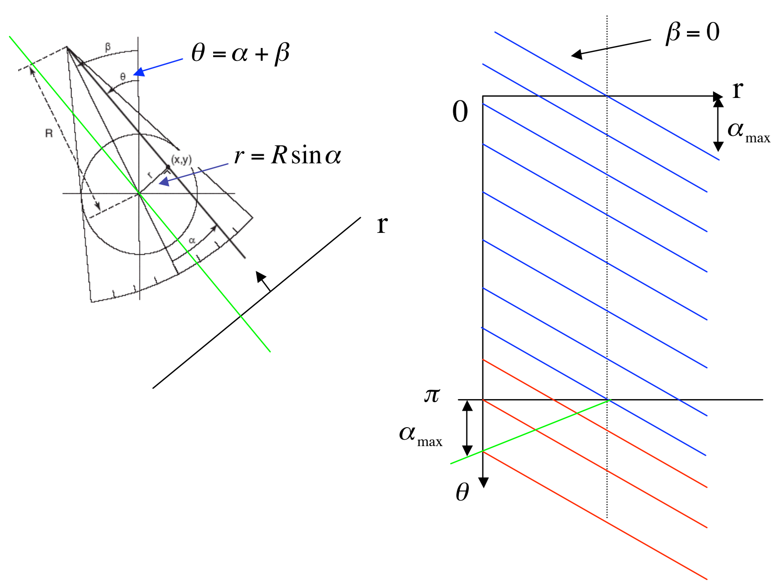
# Fan Beam

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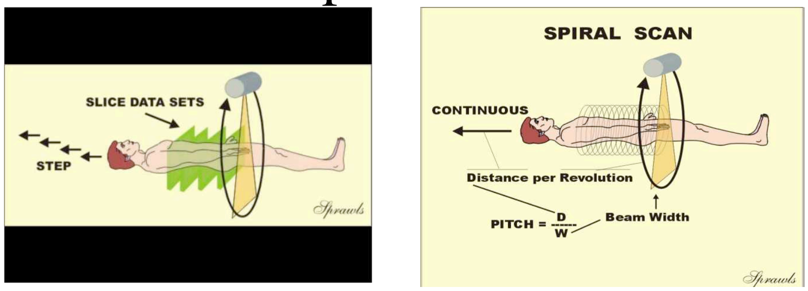
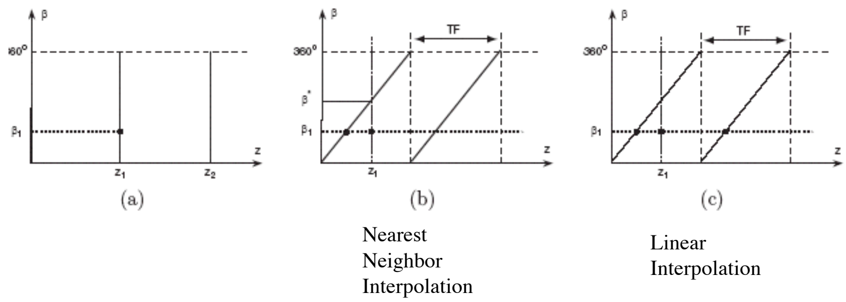
## Spiral CT

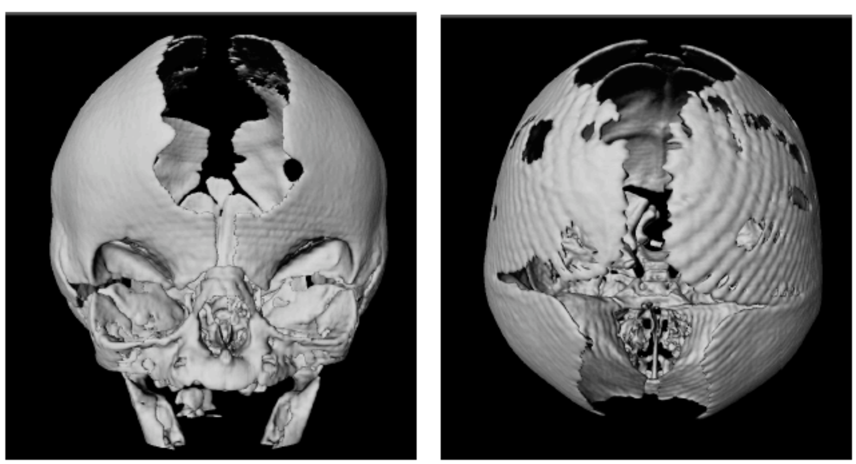



TT Liu, BE280A, UCSD Fall 2004

From http://www.sprawls.org/resources/CTIMG/classroom.htm
Suetens 2002

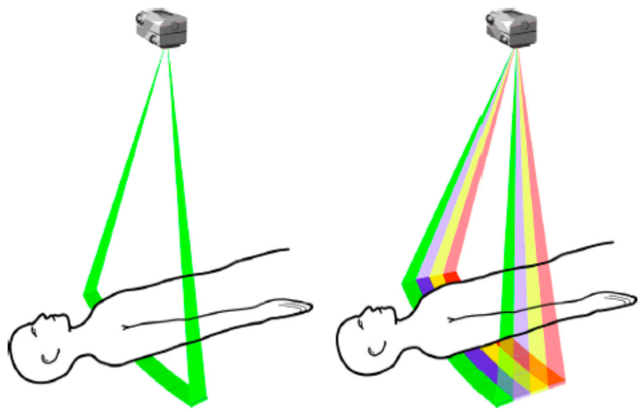
## Longitudinal Aliasing in Spiral CT



TT Liu, BE280A, UCSD Fall 2004

From http://www.sprawls.org/resources/CTIMG/classroom.htm
Suetens 2002

# Multislice CT



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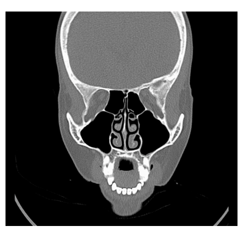
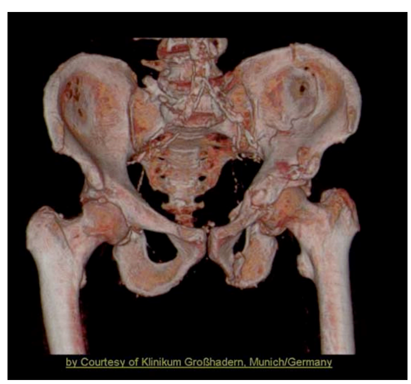
# CT Applications





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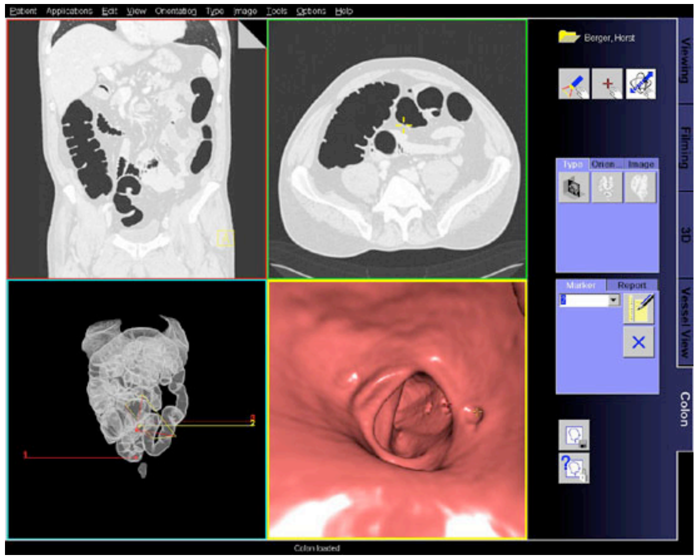
# Virtual Colonoscopy






Suetens 2002