Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
X-Rays/CT Lecture 2

Topics

• Review
• Filtered Backprojection
• Fan Beam
• Spiral CT
• Applications
\[ I_\theta(r) = I_0 \exp \left( -\int_{L_{r,\theta}} \mu(x,y) ds \right) \]

\[ = I_0 \exp \left( -\int_{L_{r,\theta}} \mu(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) ds \right) \]

Projections

\[ p_\theta(r) = -\ln \frac{I_\theta(r)}{I_0} = \int_{L_{r,\theta}} \mu(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) ds \]

Sinogram
Sinogram

![Sinogram](image)

Direct Inverse Approach

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$p_1 = \mu_1 + \mu_2$</td>
<td>$p_2 = \mu_3 + \mu_4$</td>
<td>$p_3 = \mu_1 + \mu_3$</td>
<td>$p_4 = \mu_2 + \mu_4$</td>
</tr>
</tbody>
</table>

$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$

4 equations, 4 unknowns.
Are these the correct equations to use?
Direct Inverse Approach

\[
\begin{array}{cc}
\mu_1 & \mu_2 \\
\mu_3 & \mu_4 \\
\end{array}
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
\end{array}
\begin{array}{c}
p_1 = \mu_1 + \mu_2 \\
p_2 = \mu_3 + \mu_4 \\
p_3 = \mu_1 + \mu_3 \\
p_4 = \mu_1 + \mu_4 \\
\end{array}
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{array}{c}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array}
\]

4 equations, 4 unknowns. These are linearly independent now.
In general for a NxN image, N² unknowns, N² equations.
This requires the inversion of a N²xN² matrix
For a high-resolution 512x512 image, N²=262144 equations.
Requires inversion of a 262144x262144 matrix!
Inversion process sensitive to measurement errors.

Iterative Inverse Approach

Algebraic Reconstruction Technique (ART)
### Backprojection

Given a 3x3 matrix:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
$$

The backprojection is calculated as follows:

$$
b(x, y) = B\{p(r, \theta)\}
= \int_0^\pi p(x \cos \theta + y \sin \theta, \theta) d\theta
$$

$$
b(x_0, y) = p(r, \theta = 0) \Delta \theta = p(x_0) \Delta \theta
$$
Backprojection

\[ b(x, y) = B\{p(r, \theta)\} = \int_0^\pi p(x \cos \theta + y \sin \theta, \theta) d\theta \]
**Projection Theorem**

\[ U(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x,y)e^{-j2\pi(k_xx + k_yy)} \, dx \, dy = F_{2D}[\mu(x,y)] \]

\[ U(k_x, k_y) = P(k, \theta) \]

\[ k_x = k \cos \theta \]

\[ k_y = k \sin \theta \]

\[ k = \sqrt{k_x^2 + k_y^2} \]

**Fourier Reconstruction**

Interpolate onto Cartesian grid then take inverse transform
Fourier Interpretation

Density $\approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by $|k|$ before inverse transforming.

Polar Version of Inverse FT

\[
\mu(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k_x, k_y) e^{j2\pi(k_x x + k_y y)} \, dk_x \, dk_y
\]

\[
= \int_{0}^{2\pi} \int_{0}^{\infty} U(k, \theta) e^{j2\pi(k \cos \theta + k \sin \theta)} \, k \, dk \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{-\infty}^{\infty} U(k, \theta) e^{j2\pi(x \cos \theta + y \sin \theta)} |k| \, dk \, d\theta
\]
Filtered Backprojection

\[ \mu(x, y) = \int_0^\pi \int_{-\infty}^\infty U(k, \theta) e^{j2\pi(xk \cos \theta + yk \sin \theta)} |k| dk d\theta \]

\[ = \int_0^\pi \int_{-\infty}^\infty |k| U(k, \theta) e^{j2\pi kr} dk d\theta \]

\[ = \int_0^\pi u^*(r, \theta) d\theta \quad \text{Backproject a filtered projection} \]

where \( r = x \cos \theta + y \sin \theta \)

\[ u^*(r, \theta) = \int_{-\infty}^\infty |k| U(k, \theta) e^{j2\pi kr} dk \]

\[ = u(r, \theta) \ast F^{-1}[k] \]

\[ = u(r, \theta) \ast q(r) \]

Reconstruction Path
Ram-Lak Filter

\[ q(r) = F^{-1}[k] = \int_{-\infty}^{\infty} |k| e^{i2\pi kr} \, dk \]

Not a realistic convolution kernel.

\[ q(r) = F^{-1} \left[ k \text{rect} \left( \frac{k}{2k_{\max}} \right) \right] = \int_{-\infty}^{\infty} |k| e^{i2\pi kr} \, dk \]

\[ k_{\max} = \frac{1}{\Delta s} \]

Reconstruction Path

\[ \text{Projection} \]

\[ F \]

\[ x \]

\[ F^{-1} \text{ Filtered Projection} \rightarrow \text{Back-Project} \]
Reconstruction Path

Projection

Filtered Projection

Back-Project

Example

TT Lin, BE280A, UCSD Fall 2004 Suetsens 2002

TT Lin, BE280A, UCSD Fall 2004 Kak and Slaney
Additional Filtering

Figure 5.12: (a) Hamming window with $\alpha = 0.54$ and Hamming window (dashed) with $\alpha = 0.5$. (b) Ramp filter and its products with a Hamming window and a Hamming window (dashed).

$$k_{\text{max}} = 1/\Delta s$$

Sampling Requirements

How many detectors do we need?

How many angular views do we need?
Artifacts

Object

Effect of Noise

Aliasing due to insufficient number of detectors

Aliasing due to insufficient number of views
Projection Aliasing

Aliased Image

Alien Component

(a)  (b)

(aliased image)-
(alias component)
Alias components

\[ W = \frac{2}{\Delta s} \]
\[ \delta = 1/W = \Delta s/2 \]

Sampling Requirements

Projection

Beam Width

Smoothed Projection

\[ W = \frac{2}{\Delta s} \]
\[ \delta = 1/W = \Delta s/2 \]
Detector Sampling Requirements

Beamwidth of detector $\Delta s$

Sampling interval $\Delta r$

Requirement is $\Delta r \leq \Delta s/2$
View Aliasing

View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

\[
\frac{\pi \text{FOV}}{N_{\text{views}}} = \Delta r
\]

\[
N_{\text{views,360}} = \frac{\pi \text{FOV}}{\Delta r} = \pi N_{\text{proj}} \quad \text{(for 360 degrees)}
\]

\[
N_{\text{views,180}} = \frac{\pi N_{\text{proj}}}{2} \quad \text{(for 180 degrees)}
\]
CT System Generations

(a) 5 minutes/slice
(b) 20 seconds /slice
(c) 0.5 seconds /slice

CT System

(a) (b) (c) (d)
Fan Beam

\[ \theta = \alpha + \beta \]

\[ r = R \sin \alpha \]

TT Liu, BE280A, UCSD Fall 2004

Suetens 2002
Spiral CT

Longitudinal Aliasing in Spiral CT
Multislice CT

CT Applications
Virtual Colonoscopy

TT Liu, BE280A, UCSD Fall 2004
Suetsens 2002