Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to $B_z$ such that $B_z(x,y,z) = B_0 + \Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.
**MRI System**

Simplified Drawing of Basic Instrumentation.

Body lies on table encompassed by coils for static field $B_0$, gradient fields (two of three shown), and radiofrequency field $B_1$.

Image, caption: copyright Nishimura, Fig. 3.15

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**Z Gradient Coil**

Credit: Buxton 2002
Gradient Fields

\[ B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z \]

\[ = B_0 + G_x x + G_y y + G_z z \]

\( G_z = \frac{\partial B_z}{\partial z} > 0 \)

\( G_y = \frac{\partial B_z}{\partial y} > 0 \)

Interpretation

\[ \Delta B_x(x) = G_x x \]

Spins Precess at \( \gamma B_0 - \gamma G_x x \) (slower)

Spins Precess at \( \gamma B_0 \) (faster)

Spins Precess at \( \gamma B_0 + \gamma G_x x \) (faster)
Gradient Fields

Define

\[ \mathbf{G} = G_x \mathbf{i} + G_y \mathbf{j} + G_z \mathbf{k} \]
\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]

So that

\[ G_x x + G_y y + G_z z = \mathbf{G} \cdot \mathbf{r} \]

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by:

\[ B_z(\mathbf{r}, t) = B_0 + \mathbf{G}(t) \cdot \mathbf{r} \]

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

\[ M(t) = M(0) e^{-j\omega_0 t} e^{-t/T_2} \]

In the presence of non time-varying gradients we have

\[ M(\mathbf{r}) = M(\mathbf{r}, 0) e^{-j\gamma B_z(\mathbf{r})} e^{-t/T_2(\mathbf{r})} \]
\[ = M(\mathbf{r}, 0) e^{-j\gamma (B_0 + \mathbf{G} \cdot \mathbf{r})} e^{-t/T_2(\mathbf{r})} \]
\[ = M(\mathbf{r}, 0) e^{-j\omega_0 t} e^{-j\gamma \mathbf{G} \cdot \mathbf{r}} e^{-t/T_2(\mathbf{r})} \]
**Time-Varying Gradient Fields**

In the presence of time-varying gradients the frequency as a function of space and time is:

\[ \omega(\vec{r}, t) = \gamma B_z(\vec{r}, t) \]

\[ = \gamma B_0 + \gamma \vec{G}(t) \cdot \vec{r} \]

\[ = \omega_0 + \Delta \omega(\vec{r}, t) \]

**Phase**

Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

\[ \varphi(\vec{r}, t) = -\int_0^t \omega(\vec{r}, \tau) d\tau \]

\[ = -\omega_0 t + \Delta \varphi(\vec{r}, t) \]

Where the incremental phase due to the gradients is

\[ \Delta \varphi(\vec{r}, t) = -\int_0^t \Delta \omega(\vec{r}, \tau) d\tau \]

\[ = -\int_0^t \gamma \vec{G}(\vec{r}, \tau) \cdot \vec{r} d\tau \]
Phase with constant gradient

\[ \Delta \varphi(\vec{r}, t_1) = -\int_0^{t_1} \Delta \omega(\vec{r}, \tau) d\tau \]

\[ \Delta \varphi(\vec{r}, t_2) = -\int_0^{t_2} \Delta \omega(\vec{r}, \tau) d\tau \]

\[ = -\Delta \omega(\vec{r}) t_2 \]

if \( \Delta \omega \) is non-time varying.

Phase with time-varying gradient

\[ \Delta \varphi(\vec{r}, t_1) = -\int_0^{t_1} \Delta \omega(\vec{r}, \tau) d\tau \]

\[ \Delta \varphi(\vec{r}, t_2) = -\int_0^{t_2} \Delta \omega(\vec{r}, \tau) d\tau \]

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Time-Varying Gradient Fields

The transverse magnetization is then given by

\[
M(\vec{r}, t) = M(\vec{r}, 0) e^{-t/T_2(\vec{r})} e^{j\phi(\vec{r}, t)}
\]

\[
= M(\vec{r}, 0) e^{-t/T_2(\vec{r})} e^{-j\gamma t} \exp\left(-j \int_0^t \Delta \omega(\vec{r}, \tau) d\tau\right)
\]

\[
= M(\vec{r}, 0) e^{-t/T_2(\vec{r})} e^{-j\omega_0 t} \exp\left(-j \gamma \int_0^t G(\tau) \cdot \vec{r} d\tau\right)
\]

Signal Equation

Signal from a volume

\[
s_v(t) = \int V M(\vec{r}, t) dV
\]

\[
= \int_x \int_y \int_z M(x, y, z, 0) e^{-t/T_2(\vec{r})} e^{-j\omega_0 t} \exp\left(-j \gamma \int_0^t G(\tau) \cdot \vec{r} d\tau\right) dx dy dz
\]

For now, consider signal from a slice along \( z \) and drop the \( T_2 \) term. Define

\[
m(x, y) = \int_{\Delta z/2}^{\Delta z/2} M(\vec{r}, t) dz
\]

To obtain

\[
s_s(t) = \int_x \int_y m(x, y) e^{-j\omega_0 t} \exp\left(-j \gamma \int_0^t G(\tau) \cdot \vec{r} d\tau\right) dx dy
\]
Signal Equation

Demodulate the signal to obtain

\[ s(t) = e^{j\omega_0 t}s_r(t) \]
\[ = \int_x \int_y m(x, y) \exp\left(-j\int_0^\infty \vec{G}(\tau) \cdot \vec{r} \, d\tau\right) dx dy \]
\[ = \int_x \int_y m(x, y) \exp\left(-j\int_0^\infty [G_x(\tau)x + G_y(\tau)y] \, d\tau\right) dx dy \]
\[ = \int_x \int_y m(x, y) \exp\left(-j2\pi(k_x(t)x + k_y(t)y)\right) dx dy \]

Where

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^\infty G_x(\tau) \, d\tau \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int_0^\infty G_y(\tau) \, d\tau \]

MR signal is Fourier Transform

\[ s(t) = \int_x \int_y m(x, y) \exp\left(-j2\pi(k_x(t)x + k_y(t)y)\right) dx dy \]
\[ = M(k_x(t), k_y(t)) \]
\[ = F[m(x, y)]_{k_x(t), k_y(t)} \]
**K-space**

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M(k_x(t), k_y(t)) = F[m(x, y)]_{k_x(t), k_y(t)} \]

evaluated at the spatial frequencies:

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau \]

\[ k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

**Interpretation**

\[ \exp\left(-j2\pi \frac{0}{8\Delta x} x\right) \]

\[ \exp\left(-j2\pi \frac{1}{8\Delta x} x\right) \]

\[ \exp\left(-j2\pi \frac{2}{8\Delta x} x\right) \]

\[ \Delta B_x(x) = G_x x \]

Slower           Faster

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\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau \]
K-space trajectory

Spin-Warp

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Spin-Warp

$G_x(t)$

$G_y(t)$

Spin-Warp Pulse Sequence

RF

$G_x(t)$

$G_y(t)$

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Units

Spatial frequencies \((k_x, k_y)\) have units of 1/distance. Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/distance. Most commonly G/cm or mT/m

\(\gamma/(2\pi)\) has units of Hz/G or Hz/Tesla.

\[
k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau
\]

\[
= \left[ \frac{Hz}{Gauss} \right] \left[ \frac{Gauss}{cm} \right] \left[ sec \right]
\]

\[
= \left[ \frac{1}{cm} \right]
\]

Example

\(G_x(t) = 1\) Gauss/cm

\[k_x(t_1) \quad k_x(t_2)\]

\[k_y\]

\(t_2 = 0.235\) ms

\[
k_x(t_2) = \frac{\gamma}{2\pi} \int_0^{t_2} G_x(\tau) d\tau
\]

\[
= 4257\text{Hz/G}\cdot1G/cm\cdot0.235\times10^{-3}\text{s}
\]

\[
= 1\text{cm}^{-1}
\]

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