Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2005
Noise and Estimation

What is Noise?

Fluctuations in either the imaging system or the object being imaged.

Quantization Noise: Due to conversion from analog waveform to digital number.

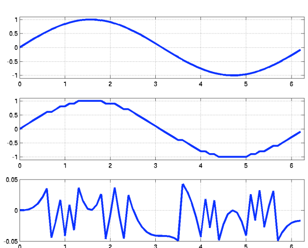
Quantum Noise: Random fluctuation in the number of photons emittted and recorded.

Thermal Noise: Random fluctuations present in all electronic systems. Also, sample noise in MRI

Other types: flicker, burst, avalanche - observed in semiconductor devices.

Structured Noise: physiological sources, interference

Quantization Noise



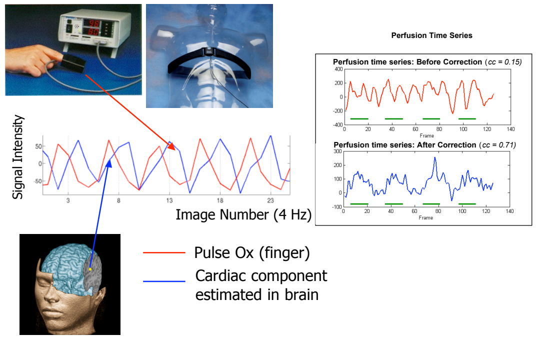
Signal s(t)

Quantized Signal r(t)

Quantization noise q(t)

r(t) = s(t) + q(t)
Although the noise is deterministic, it is useful to model the noise as a random process.

Physiological Noise



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Noise and Image Quality

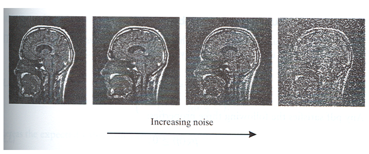

Figure 3.10
The effect of noise on image quality: image quality decreases rapidly with increasing noise contamination.

Prince and Links 2005

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Thermal Noise

Fluctuations in voltage across a resistor due to random thermal motion of electrons.

Described by J.B. Johnson in 1927 (therefore sometimes called Johnson noise). Explained by H. Nyquist in 1928.

$$\langle V^2 \rangle = 4kT \cdot R \cdot BW$$

Variance in Voltage

Temperature

Resistance

Bandwidth

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Thermal Noise

$\langle V^2 \rangle = 4kT \cdot R \cdot BW$

At room temperature, noise in a 1 kΩ resistor is
$\langle V^2 \rangle / BW = 16 \times 10^{-18}\ V^2/Hz$

In root mean squared form, this corresponds to
$V/BW = 4\ nV/\sqrt{Hz}$.

Example: For BW = 250 kHz and 2 kΩ resistor, total noise voltage is
$\sqrt{2 \cdot 16 \times 10^{-18} \cdot 250 \times 10^3} = 4\ \mu V$

Thermal Noise

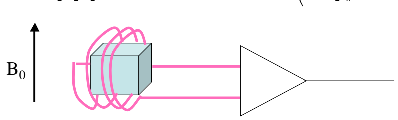
Noise spectral density is independent of frequency up to 10^{13} Hz. Therefore it is a source of white noise.

Amplitude distribution of the noise is Gaussian.

Signal in MRI

Recall the signal equation has the form

$$s_r(t) = \int\int\int M(x,y,z) e^{-t/T_2(\mathbf{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \mathbf{G}(\tau) \cdot \mathbf{r}(\tau) d\tau\right) dxdydz$$



Faraday's Law

$$EMF = -\frac{\partial \phi}{\partial t}$$

$$\phi = \text{Magnetic Flux} = \int \mathbf{B}_1(x,y,z) \cdot \mathbf{M}(x,y,z) dV$$

Signal in MRI

Signal in the receiver coil

$$s_r(t) = j\omega_0 B_{1xy} \int M(x,y,z) e^{-t/T_2(\mathbf{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \mathbf{G}(\tau)\cdot \mathbf{r}(\tau) d\tau\right) dV$$

Recall, total magnetization is proportional to B_0

Also $\omega_0 = \gamma B_0$.

Therefore, total signal is proportional to B_0^2

Noise in MRI

Primary sources of noise are:
1) Thermal noise of the receiver coil
2) Thermal noise of the sample.

Coil Resistance: At higher frequencies, the EM waves tend to travel along the surface of the conductor (skin effect). As a result,

$$R_{\text{coil}} \propto \omega_0^{1/2} \Rightarrow \langle N_{coil}^2 \rangle \propto \omega_0^{1/2} \propto B_0^{1/2}$$

Sample Noise: Noise is white, but differentiation process due to Faraday's law introduces a multiplication by ω_0. As a result, the noise variance from the sample is proportional to ω_0^2.

$$\langle N_{sample}^2 \rangle \propto \omega_0^2 \propto B_0^2$$

SNR in MRI

$$SNR = \frac{\text{signal amplitude}}{\text{standard deviation of noise}} \propto \frac{B_0^2}{\sqrt{\alpha B_0^{1/2} + \beta B_0^2}}$$

If coil noise dominates

$$SNR \propto B_0^{7/4}$$

If sample noise dominates

$$SNR \propto B_0$$

Random Variables

A random variable X is characterized by its cumulative distribution function (CDF)

$\Pr(X \leq x) = F_X(x)$

The derivative of the CDF is the probability density function(pdf)

$f_X(x) = dF_X(x)/dx$

The probability that X will take on values between two limits x_1 and x_2 is

$\Pr(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$

Mean and Variance

$$\begin{aligned} \mu_X &= E[X] \\ &= \int_{-\infty}^{\infty} x f_X(x)dx \end{aligned}$$

$$\begin{aligned} \sigma_X^2 &= Var[X] \\ &= E[(X-\mu_X)^2] \\ &= \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx \\ &= E[X^2] - \mu_X^2 \end{aligned}$$

Gaussian Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/\left(2\sigma^2\right)\right)$$

$$\begin{aligned} \mu_X &= \mu \\ \sigma_X^2 &= \sigma^2 \end{aligned}$$

Independent Random Variables

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$$
$$E[X_1X_2] = E[X_1]E[X_2]$$

$$\begin{aligned}
&\text{Let } Y = X_1 + X_2 \text{ then} \\
&\mu_Y = E[Y] \\
&\quad = E[X_1] + E[X_2] \\
&\quad = \mu_1 + \mu_2 \\
&E[Y^2] = E[X_1^2] + 2E[X_1]E[X_2] + E[X_2^2] = E[X_1^2] + 2\mu_1\mu_2 + E[X_2^2] \\
&\sigma_Y^2 = E[Y^2] - \mu_Y^2 \\
&\quad = E[X_1^2] + 2\mu_1\mu_2 + E[X_2^2] - \mu_1^2 - \mu_2^2 - 2\mu_1\mu_2 \\
&\quad = \sigma_{X_1}^2 + \sigma_{X_2}^2
\end{aligned}$$

Signal Averaging

We can improve SNR by averaging.

Let

$y_1 = y_0 + n_1$

$y_2 = y_0 + n_2$

The sum of the two measurements is $2y_0 + (n_1 + n_2)$.

If the noise in the measurements is independent, then the variances sum and the total variance is $2\sigma_n^2$

$$SNR_{Tot} = \frac{2y_0}{\sqrt{2}\sigma_n} = \sqrt{2}SNR_{original}$$

In general, $SNR \propto \sqrt{N_{ave}} \propto \sqrt{Time}$

Random Processes

A random process is an indexed family of random variables

Examples:
discrete: $X_1, X_2, \ldots, X_N$
continuous: $X(t)$

If all the random variables share the same pdf and take on values independently, the process is said to be independent and identically distributed (iid).

Example: unbiased coin tosses

If the joint statistics of the process do not vary with index, the process is said to be stationary.

Correlation and Covariance

Correlation

$R(t_1,t_2) = E(X(t_1)X^*(t_2))$

$R(i,j) = E(X_i X_j)$

Covariance

$C(t_1,t_2) = E\Big(\big(X(t_1) - \overline{X}(t_1)\big)\big(X(t_2) - \overline{X}(t_2)\big)^*\Big)$

$C(i,j) = E\Big(\big(X_i - \overline{X}_i\big)\big(X_j - \overline{X}_j\big)\Big)$

Stationary Process

For a wide - sense stationary process

$E(X(t)) = \mu$

$R(t,t_2) = R(\tau) = E(X(t)X^*(t+\tau))$ for $\tau = t_2 - t$

$R(i,j) = R(m) = E(X_i X_{i+m})$ for $m = j - i$

Example: White noise process

$E[X] = 0$

$C(\tau) = \sigma^2\delta(\tau)$

Power Spectral Density

For a wide - sense stationary process, we can define the Power Spectral Density as:

$S_X(f) = F\{R(\tau)\}$ for a continuous random process

or

$S_X(f) = F\{R(m)\}$ for a discrete random process

Example: White noise

$S_X(f) = F\{\sigma^2\delta(\tau)\} = \sigma^2$

Vector Notation

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

$$\mathbf{R} = E(\mathbf{X}\mathbf{X}^H) = E\left(\begin{bmatrix} X_1X_1^* & X_1X_2^* & \cdots & X_1X_N^* \\ X_2X_1^* & X_2X_2^* & \cdots & X_2X_N^* \\ \vdots & \vdots & \ddots & \vdots \\ X_NX_1^* & X_NX_2^* & \cdots & X_NX_N^* \end{bmatrix}\right)$$

Example

$\mathbf{X}$ denotes a stationary random process with mean zero and correlation $R[m] = \sigma^2\delta[m]$

$$\mathbf{R} = E(\mathbf{X}\mathbf{X}^H) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

$$= \sigma^2\mathbf{I}$$

Noise in k-space

Recall that in MRI we acquire samples in k-space. The noise in these samples is typically well described by an iid random process.
For Cartesian sampling, the noise in the image domain is then also described by an iid random process.

For each point in k-space, $SNR = \frac{S(k)}{\sigma_n}$ where $S(k)$ is the signal and σ_n is the standard deviation of each noise sample.

Noise in image space

Noise variance per sample in k - space is σ_n^2.
Each voxel in image space is obtained from the Fourier transform of k - space data.
Say there are N points in k - space. The overall noise variance contribution of these N points is $N\sigma_n^2$.
If we assume a point object, then all points in k - space contribute equally to the signal, so overall signal is NS_0.
Then overall SNR in image space is

$$SNR \propto \frac{NS_0}{\sqrt{N\sigma_n}} = \sqrt{N}\frac{S_0}{\sigma_n}$$

Therefore, SNR increases as we increase the matrix size.

Signal Averaging

We can improve SNR by averaging in k - space
In general, $SNR \propto \sqrt{N_{ave}} \propto \sqrt{Time}$

Effect of Readout Window

ADC samples acquired with sampling period Δt.
Thermal noise per sample is $\sigma_n^2 \propto \Delta f = \frac{1}{\Delta t}$

If we double length of the readout window, the noise variance per sample decreases by two.

The noise standard deviation decreases by $\sqrt{2}$, and the SNR increases by $\sqrt{2}$.

In general, SNR $\propto \sqrt{T_{\mathrm{Read}}} = \sqrt{N_{k_x}\Delta t}$

SNR and Phase Encodes

Assume that spatial resolution is held constant. What happens if we increase the number of phase encodes? Recall that $\delta_y = \frac{1}{W_{k_y}}$. Thus, increasing the number of phase encodes N_{PE}, decreases Δk_y and increases FOV_y.

If we double the number of phase encodes, each point in image space has double the number of k - space lines contributing to its signal. The noise variances sum. The SNR therefore goes up by $\sqrt{2}$.

In general $SNR \propto \sqrt{N_{PE}}$

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Overall SNR

$$SNR \propto \frac{Signal}{\sigma_n} \propto \frac{\Delta x \Delta y \Delta z}{\sigma_n}$$

Putting everything together, we find that

$$SNR \propto \sqrt{N_{ave} N_x N_{PE} \Delta t}\,\Delta x \Delta y \Delta z$$

$$= \sqrt{Measurement\ Time} \cdot Voxel\ Volume$$

In general,

$$SNR \propto \sqrt{Measurement\ Time} \cdot Voxel\ Volume \cdot f(\rho, T_1, T_2)$$

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Example

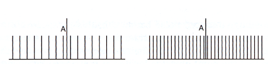

Figure 7.18: Example: 256 readout samples vs 512 readout samples.

Assume same readout gradients for both sequences. Sampling rate for sequence 2 is twice that of sequence 1. What are the relative SNRs?

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Nishimura 1996

Example

Sampling rate for sequence 2 is twice as large, so that bandwidth is doubled. Therefore noise variance is also doubled

$$SNR1 = \frac{256A}{\sqrt{256\sigma_n^2}} = \frac{\sqrt{256}A}{\sigma_n}$$

$$SNR1 = \frac{512A}{\sqrt{512\left(2\sigma_n^2\right)}} = \frac{\sqrt{512}A}{\sqrt{2}\sigma_n} = \frac{\sqrt{256}A}{\sigma_n}$$

Note that sequences have the same resolution, but sequence 2 has twice the FOV.