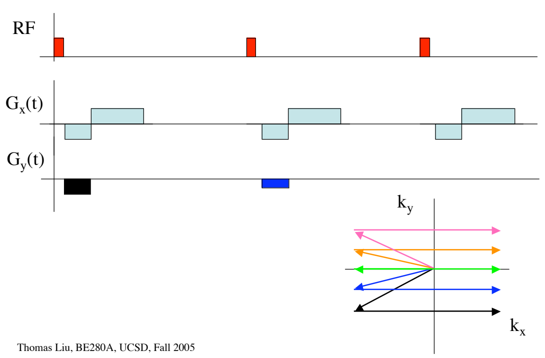
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2005
Linear Systems Lecture 1

Thomas Liu, BE280A, UCSD, Fall 2005

Spin-Warp Pulse Sequence



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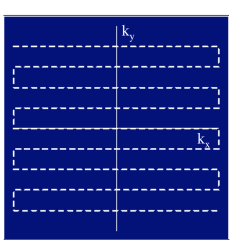
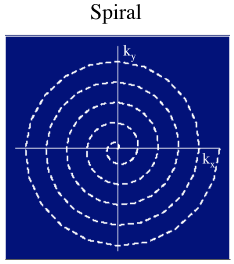
K-space trajectories

EPI



Spiral



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Credit: Larry Frank

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

evaluated at the spatial frequencies:

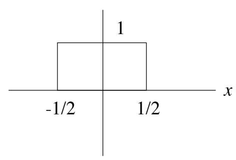
$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

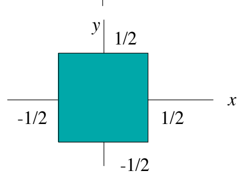
Rectangle Function

$$\Pi(x) \; = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$



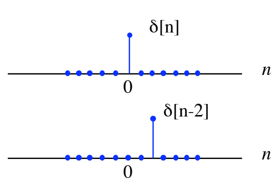
Also called rect(x)

$$\Pi(x,y) = \; \Pi(x)\Pi(y)$$



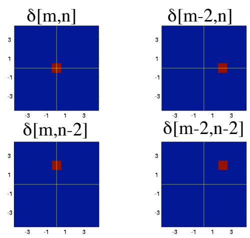
Kronecker Delta Function

$$\delta[n] \; = \begin{cases} 1 & \text{for } n = 0 \\ 0 & otherwise \end{cases}$$



Kronecker Delta Function

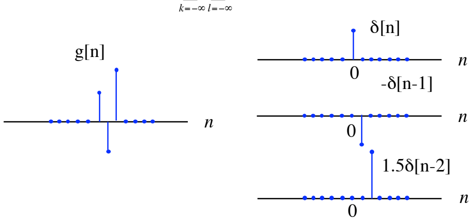
$$\delta[m,n] = \begin{cases} 1 & \text{for } m = 0, n = 0 \\ 0 & \textit{otherwise} \end{cases}$$



Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or $^2\delta(x,y)$ - 2D Dirac Delta Function

$\delta(x,y,z)$ or $^3\delta(x,y,z)$ - 3D Dirac Delta Function

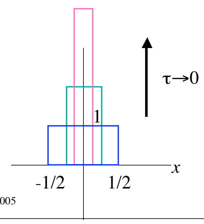
$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

1D Dirac Delta Function

$\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function

such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



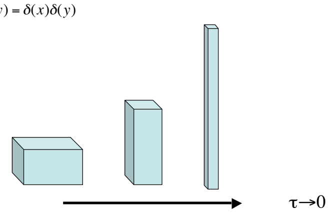
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \lim_{\tau \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-2}\Pi(x/\tau, y/\tau)dxdy$.

Useful fact : $\delta(x,y) = \delta(x)\delta(y)$

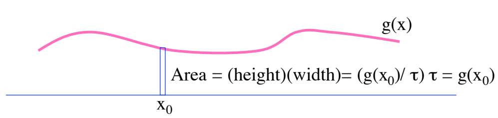


Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case

$\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)g(x)dx = g(x_0)$



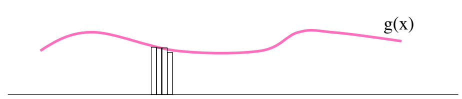
Working with Dirac Delta Functions

What is $\delta(ax - b)$? What is $d\delta(x)/dx$?
How do we define generalized functions?
There are two main approaches:
1) Look at the limit of an integral with sequences.
2) Consider the behavior of the function when integrated with a *nice* test function. Two generalized functions $\delta_1(t)$ and $\delta_2(t)$ are equivalent in the distributional sense when $\int_{-\infty}^{\infty}\delta_1(t)\phi(t)dt = \int_{-\infty}^{\infty}\delta_2(t)\phi(t)dt$

Example: $\delta(ax) = ??$

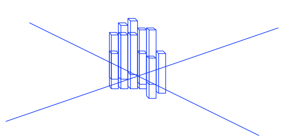
Representation of 1D Function

From the sifting property, we can write a 1D function as
$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation
$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$



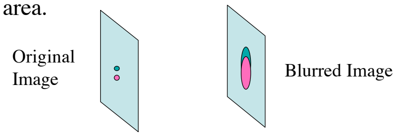
Representation of 2D Function

Similarly, we can write a 2D function as
$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi, y-\eta)d\xi d\eta.$
To gain intuition, consider the approximation
$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$



Impulse Response

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).

The Fourier Transform

The Fourier Transform (FT) is simply given by the basis coefficients

$$G(f) = \langle e^{j2\pi ft}, g(t) \rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

The Inverse Fourier Transform is the continuous-time integral expansion for $g(t)$:

$$g(t) = \int_{-\infty}^{\infty} G(f) b_f(t) df = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

This can also be written as an inner product in Fourier Space

$$g(t) = \langle e^{-j2\pi ft}, G(f) \rangle$$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \langle e^{j2\pi ft}, g(t) \rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$ Fourier Transform

$$g(t) = \langle e^{-j2\pi ft}, G(f) \rangle = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$ Inverse Fourier Transform

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \langle e^{j2\pi k_x x}, g(x) \rangle = \int_{-\infty}^{\infty} g(x) e^{-j2\pi k_x x} dx$$ Fourier Transform

$$g(x) = \langle e^{-j2\pi k_x x}, G(k_x) \rangle = \int_{-\infty}^{\infty} G(k_x) e^{j2\pi k_x x} dk_x$$ Inverse Fourier Transform

2D Fourier Transform

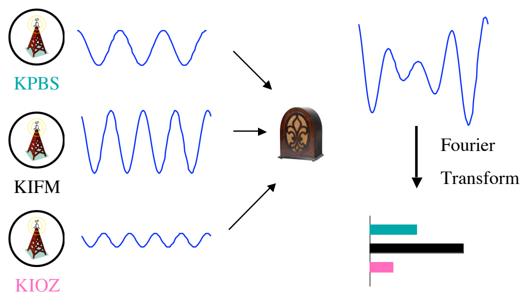
Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \left\langle e^{j2\pi(k_x x + k_y y)}, g \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(k_x x + k_y y)} dxdy$$

Inverse Fourier Transform

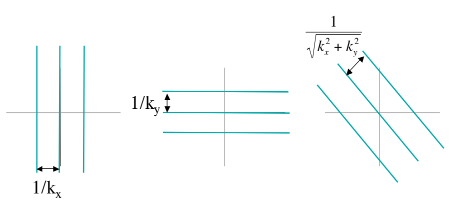
$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y$$

1D Fourier Transform



Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

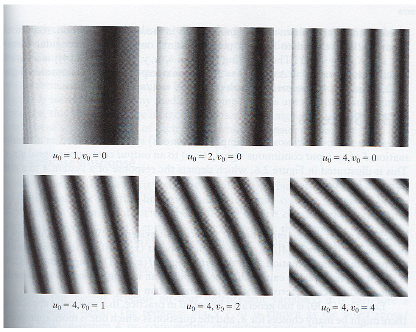
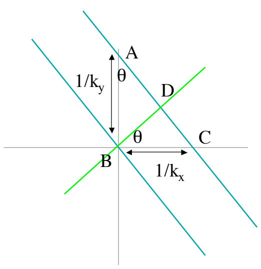


Figure 2.5 from Prince and Link

Plane Waves



$\Delta ABC \sim \Delta BDC$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2} + \frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

Basic Properties

Linearity

$$F\left[ag(x,y) + bh(x,y)\right] = aG(k_x,k_y) + bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

Shift

$$F\left[g(x-a,y-b)\right] = G(k_x,k_y)e^{-j2\pi(k_x a + k_y b)}$$

Modulation

$$F\left[g(x,y)e^{j2\pi(xa+yb)}\right] = G(k_x - a, k_y - b)$$

Computing Transforms

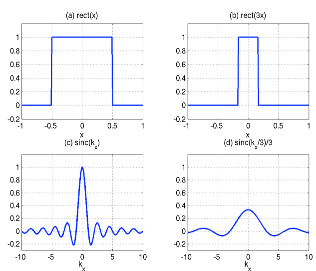
$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F(\Pi(x)) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \mathrm{sinc}(k_x) \end{aligned}$$

Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|} G\left(\frac{k_x}{a}\right)$$



Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x) g_Y(y)$
The Fourier Transform is then separable as well.

$$\begin{aligned} G(k_x,k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(k_x x + k_y y)} dx dy \\ &= \int_{-\infty}^{\infty} g_X(x) e^{-j2\pi k_x x} dx \int_{-\infty}^{\infty} g_Y(y) e^{-j2\pi k_y y} dy \\ &= G_X(k_x) G_Y(k_y) \end{aligned}$$

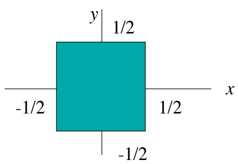
Example

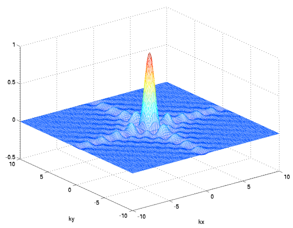
$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Example (sinc/rect)

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$





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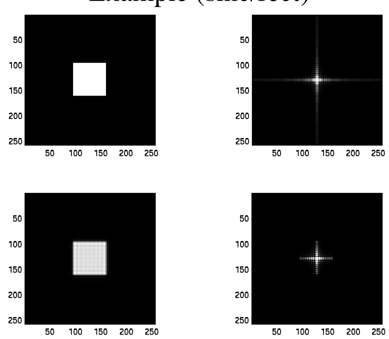
Example (sinc/rect)



Computing Transforms

$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned}\int_{-\infty}^{\infty} G(k_x)h(k_x)dk_x &= \int_{-\infty}^{\infty} G(k_x)\int_{-\infty}^{\infty} e^{-j2\pi k_x x}dxdk_x \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x)e^{-j2\pi k_x x}dk_x dx \\ &= \int_{-\infty}^{\infty} g(-x)dx \\ &= G(0)\end{aligned}$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

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Linearity

The Fourier Transform is linear.

$$F\{ag(x)+bh(x)\} = aG(k_x)+bH(k_x)$$

$$F[ag(x,y)+bh(x,y)] = aG(k_x,k_y)+bH(k_x,k_y)$$

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Computing Transforms

Similarly,

$$F\{e^{j2\pi k_0 x}\} = \delta(k_x - k_0)$$

$$F\{\cos 2\pi k_0 x\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$

$$F\{\sin 2\pi k_0 x\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$

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Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

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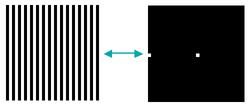
Examples

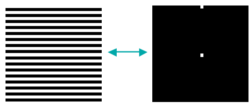
$g(x,y) = 1 + e^{-j2\pi ax}$

$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x + a)\delta(k_y)$

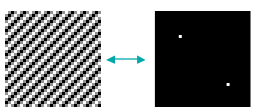


$g(x,y) = 1 + e^{j2\pi ay}$

$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x)\delta(k_y - a)$



Examples



$$g(x,y) = \cos(2\pi(ax + by))$$

$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

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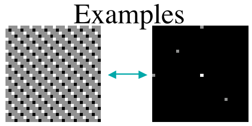
Examples



$$\begin{aligned} G(k_x,k_y) = {} & \delta(k_x,k_y) + \\ & \delta(k_x + c)\delta(k_y) + \\ & \delta(k_x)\delta(k_y - d) + \\ & \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b) \end{aligned}$$

$g(x,y) = ???$

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Duality

Note the similarity between these two transforms

$$F\left\{e^{j2\pi ax}\right\} = \delta(k_x - a)$$
$$F\left\{\delta(x-a)\right\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\left\{G(x)\right\} = g(-k_x)$$

Application of Duality

$$F\left\{\text{sinc}(x)\right\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\left\{\Pi(x)\right\} = \text{sinc}(k_x)$.

Therefore from duality, $F\left\{\text{sinc}(x)\right\} = \Pi(-k_x) = \Pi(k_x)$

Shift Theorem

$$F\left\{g(x-a)\right\} = G(k_x)e^{-j2\pi a k_x}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.

Each frequency component is shifted by a. This corresponds to a relative phase shift of

$-2\pi a/(\text{spatial period}) = -2\pi a k_x$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields $\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$

Modulation

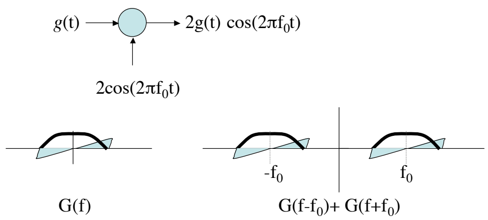
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

Example

Amplitude Modulation (e.g. AM Radio)



Modulation Example