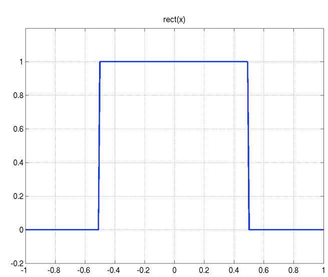
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2005
Linear Systems Lecture 3



# Fourier Sampling

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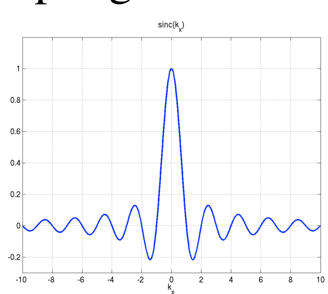
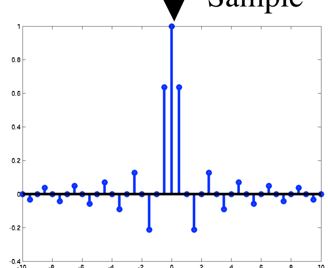


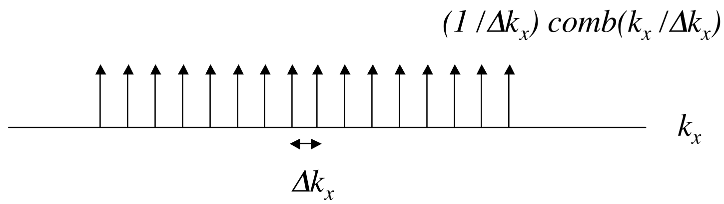

Sample



Instead of sampling the signal, we sample its Fourier Transform

??? $F^{-1}$

# Fourier Sampling



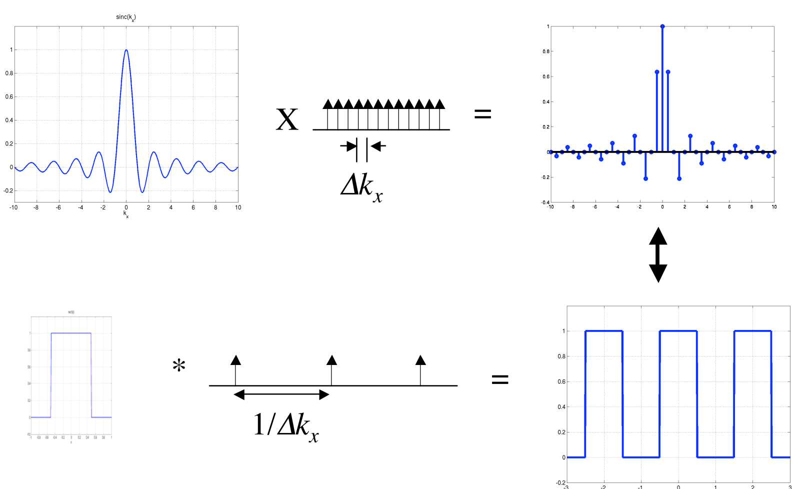

$$G_S(k_x) = G(k_x)\frac{1}{\Delta k_x}comb\left(\frac{k_x}{\Delta k_x}\right)$$

$$= G(k_x)\sum_{n=-\infty}^{\infty}\delta(k_x - n\Delta k_x)$$

$$= \sum_{n=-\infty}^{\infty}G(n\Delta k_x)\delta(k_x - n\Delta k_x)$$



# Fourier Sampling -- Inverse Transform

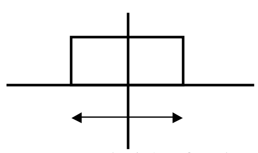


# Fourier Sampling -- Inverse Transform

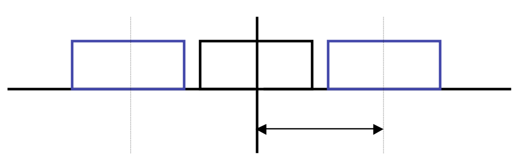
$$
\begin{aligned}
g_S(x) &= F^{-1}\left[G_S(k_x)\right] \\
&= F^{-1}\left[G(k_x)\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= F^{-1}\left[G(k_x)\right] * F^{-1}\left[\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= g(x) * comb\left(x\Delta k_x\right) \\
&= g(x) * \sum_{n=-\infty}^{\infty} \delta(x\Delta k_x - n) \\
&= g(x) * \frac{1}{\Delta k_x}\sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k_x}) \\
&= \frac{1}{\Delta k_x}\sum_{n=-\infty}^{\infty} g(x - \frac{n}{\Delta k_x})
\end{aligned}
$$



# Nyquist Condition



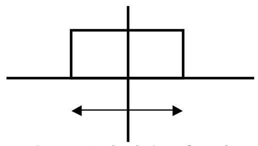
FOV (Field of View)



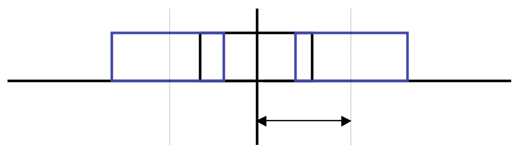
$1/\Delta k_x$

To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/\text{FOV}$

# Aliasing



FOV (Field of View)



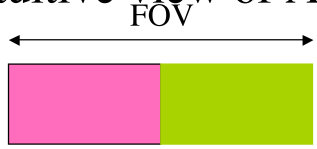
$1/\Delta k_x$

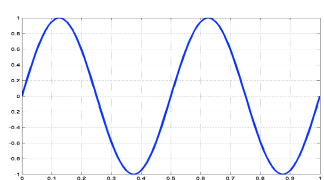
Aliasing occurs when $1/\Delta k_x <$ FOV



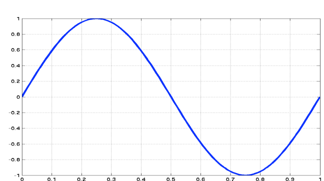
# Intuitive view of Aliasing

FOV



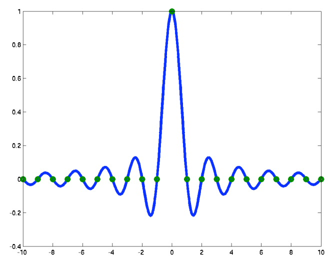


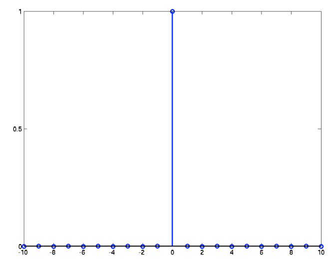
$k_x = 2/FOV$



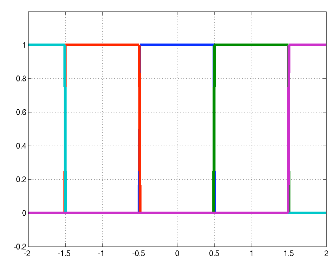
$k_x = 1/FOV$

# Aliasing Example





$\Delta k_x = 1$

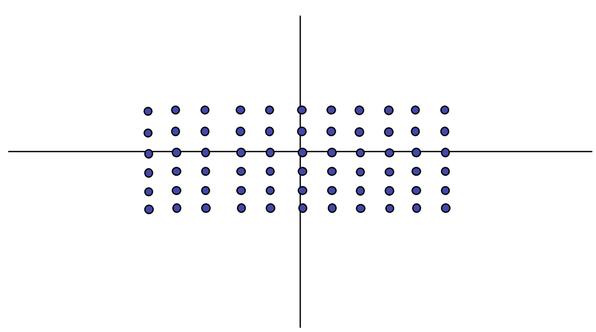


$1/\Delta k_x = 1$



# 2D Comb Function

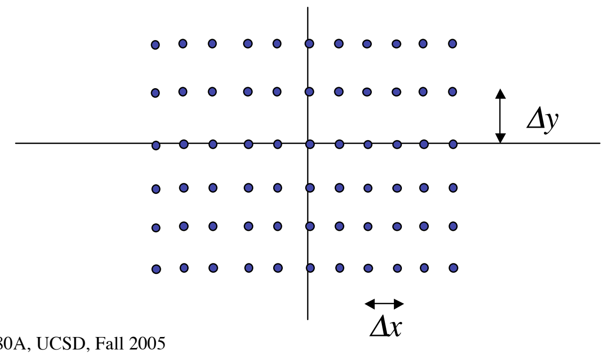
$$
\begin{aligned}
comb(x,y) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m)\delta(y-n) \\
&= comb(x)comb(y)
\end{aligned}
$$



# Scaled 2D Comb Function

$$comb(x/\Delta x, y/\Delta y) = comb(x/\Delta x)comb(y/\Delta y)$$

$$= \Delta x \Delta y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x)\delta(y - n\Delta y)$$

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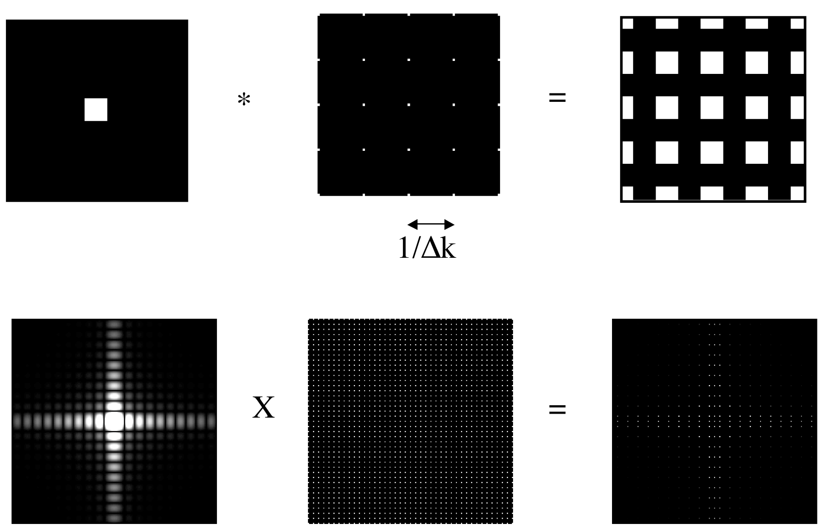


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# 2D k-space sampling

$$G_S(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)$$

$$= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

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# 2D k-space sampling

$$g_S(x,y) = F^{-1}\left[G_S(k_x,k_y)\right]$$

$$= F^{-1}\left[G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right]$$

$$= F^{-1}\left[G(k_x,k_y)\right] * F^{-1}\left[\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right]$$

$$= g(x,y) ** comb\left(x\Delta k_x\right) comb\left(y \Delta k_y\right)$$

$$= g(x) ** \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x\Delta k_x - m)\delta(y\Delta k_y - n)$$

$$= g(x) ** \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - \frac{m}{\Delta k_x})\delta(y - \frac{n}{\Delta k_y})$$

$$= \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(x - \frac{m}{\Delta k_x}, y - \frac{n}{\Delta k_y})$$

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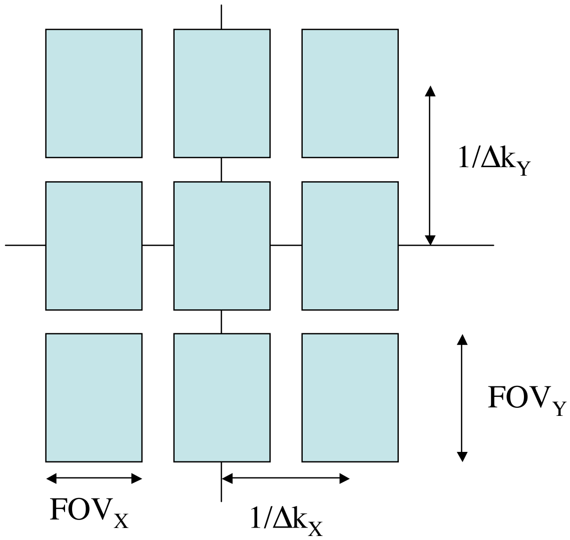
# Nyquist Conditions




$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

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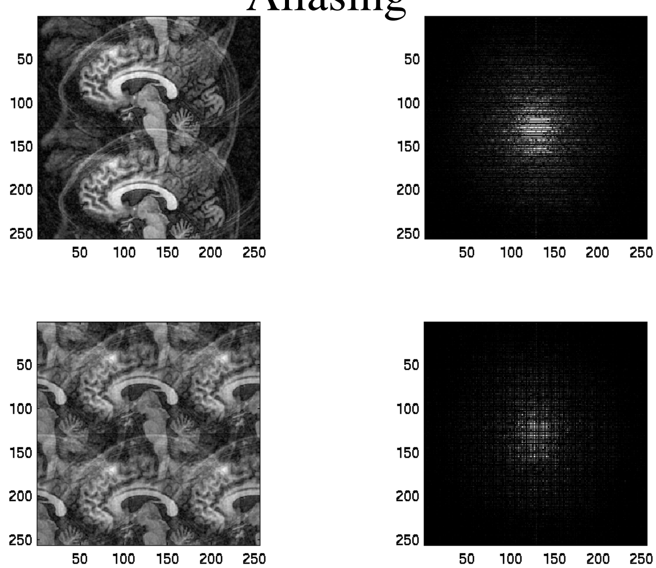
# Aliasing




Tl

# Windowing

Windowing the data in Fourier space

$$G_W(k_x,k_y) = G(k_x,k_y)W(k_x,k_y)$$

Results in convolution of the object with the inverse transform of the window

$$g_W(x,y) = g(x,y) * w(x,y)$$

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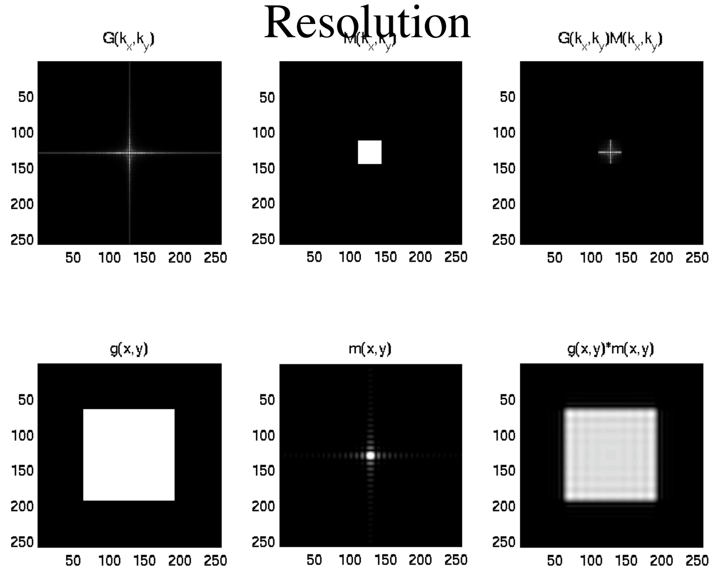
# Resolution



# Windowing Example

$$W\left(k_x,k_y\right) = rect\left(\frac{k_x}{W_{k_x}}\right) rect\left(\frac{k_y}{W_{k_y}}\right)$$

$$w\left(x,y\right) = F^{-1}\left[rect\left(\frac{k_x}{W_{k_x}}\right) rect\left(\frac{k_y}{W_{k_y}}\right)\right]$$

$$= W_{k_x} W_{k_y} \operatorname{sinc}\left(W_{k_x} x\right) \operatorname{sinc}\left(W_{k_y} y\right)$$

$$g_W\left(x,y\right) = g(x,y) ** W_{k_x} W_{k_y} \operatorname{sinc}\left(W_{k_x} x\right) \operatorname{sinc}\left(W_{k_y} y\right)$$



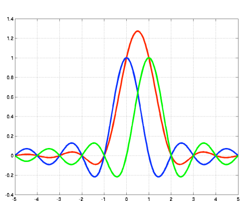
# Windowing Example

$$g(x,y) = \left[\delta(x) + \delta(x-1)\right]\delta(y)$$

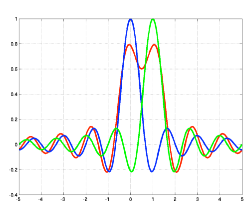
$$g_W\left(x,y\right) = \left[\delta(x) + \delta(x-1)\right]\delta(y) ** W_{k_x} W_{k_y} \operatorname{sinc}\left(W_{k_x} x\right) \operatorname{sinc}\left(W_{k_y} y\right)$$

$$= W_{k_x} W_{k_y} \left(\left[\delta(x) + \delta(x-1)\right] * \operatorname{sinc}\left(W_{k_x} x\right)\right) \operatorname{sinc}\left(W_{k_y} y\right)$$

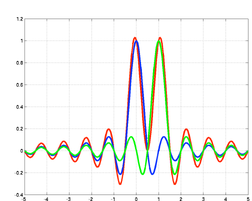
$$= W_{k_x} W_{k_y} \left(\operatorname{sinc}\left(W_{k_x} x\right) + \operatorname{sinc}\left(W_{k_x}(x-1)\right)\right) \operatorname{sinc}\left(W_{k_y} y\right)$$



$W_{k_x} = 1$



$W_{k_x} = 1.5$



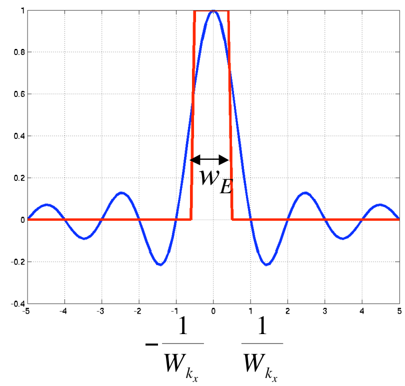
$W_{k_x} = 2$

# Effective Width

$$w_E = \frac{1}{w(0)} \int_{-\infty}^{\infty} w(x)dx$$

Example

$$w_E = \frac{1}{1}\int_{-\infty}^{\infty} \text{sinc}(W_{k_x} x)dx$$
$$= F\left[\text{sinc}(W_{k_x} x)\right]\Big|_{k_x=0}$$
$$= \frac{1}{W_{k_x}} rect\left(\frac{k_x}{W_{k_x}}\right)\Bigg|_{k_x=0}$$
$$= \frac{1}{W_{k_x}}$$



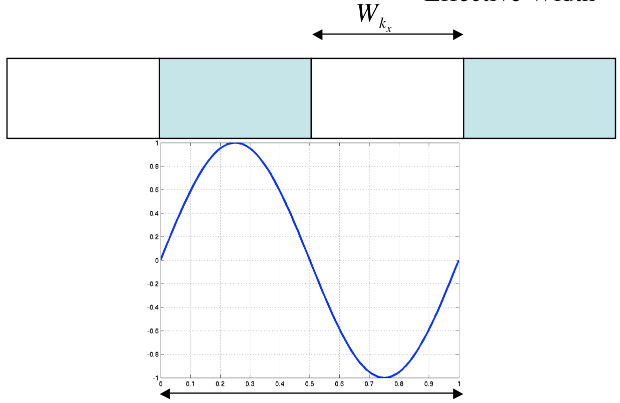



# Resolution and spatial frequency

With a window of width $W_{k_x}$ the highest spatial frequency is $W_{k_x}/2$.

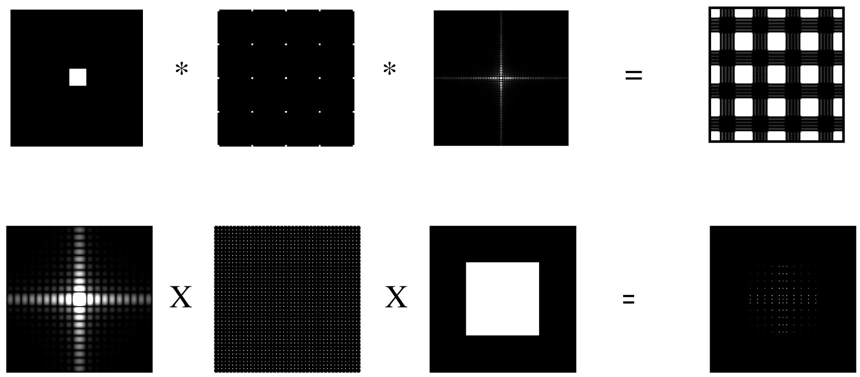
This corresponds to a spatial period of $2/W_{k_x}$.

$\frac{1}{W_{k_x}}$ = Effective Width



$\frac{2}{W_{k_x}}$

# Sampling and Windowing





# Sampling and Windowing

Sampling and windowing the data in Fourier space

$$G_{SW}\left(k_x,k_y\right) = G\left(k_x,k_y\right)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Results in replication and convolution in object space.

$$g_{SW}\left(x,y\right) = W_{k_x} W_{ky} g\left(x,y\right) ** comb(\Delta k_x x, \Delta k_y y) ** \mathrm{sinc}(W_{k_x} x)\,\mathrm{sinc}(W_{k_y} y)$$

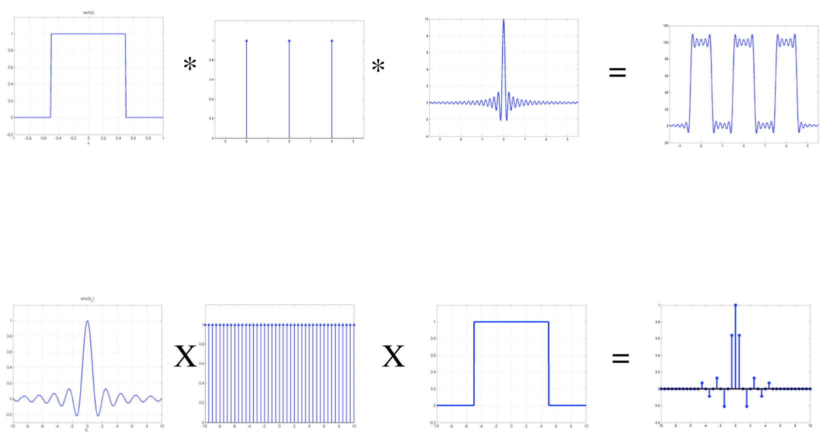
# Discrete Fourier Transform

Idea: If we sample and window in the Fourier domain, we obtain a finite number of discrete Fourier samples. When we reconstruct the object, we should have the same number of pixels in our object.

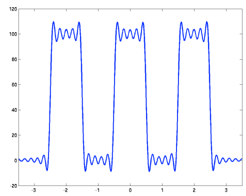
Also, the windowing process, has band-limited the sampled Fourier transform, so this allows us to sample the replicated object at discrete points.



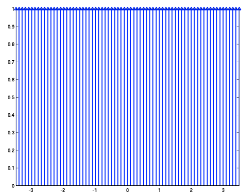
# 1D Sampling and Windowing



# Discrete Fourier Transform



x



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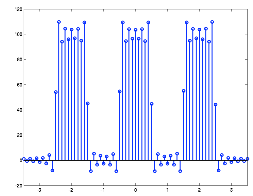

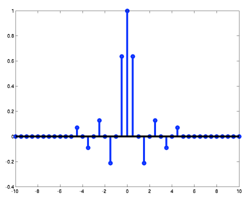

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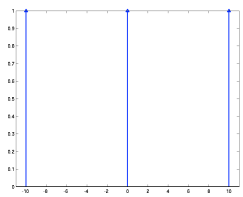

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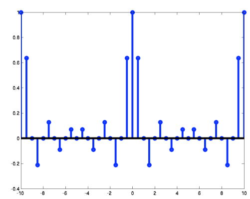
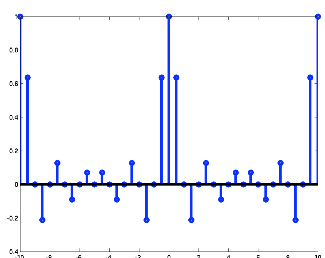
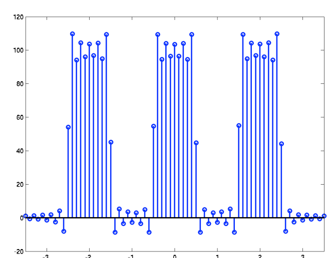




# Discrete Fourier Transform





$$W_{k_x}$$

$$FOV$$

$$N_x = \frac{W_{k_x}}{\Delta k_x}$$

$$N_x = \frac{FOV_X}{1/W_{k_x}} = \frac{W_{k_x}}{\Delta k_x}$$

Note that $\frac{FOV_X}{N_x} = \frac{1}{W_{k_x}} = \delta_x$, our measure for resolution.

# DFT

Sampling, windowing, and replication in Fourier space

$$G_{DFT}(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right) ** comb\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Replication, convolving, sampling in object space

$$g_{DFT}(x,y) = W_{k_x} W_{ky} g(x,y) ** comb(\Delta k_x x, \Delta k_y y) ** \sin c(W_{k_x} x) \sin c(W_{k_y} y)$$
$$\times W_{k_x} W_{ky} comb(W_{k_x} x, W_{k_x} y)$$

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# DFT

$$\begin{aligned}
F[g_D(x)] &= \int_0^{FOV} g_D(x) e^{-j2\pi m \Delta k_x x} dx \\
&= \int_0^{FOV} \sum_{n=-\infty}^{\infty} g_D(n/W_x)\delta(x - n/W_x) e^{-j2\pi m \Delta k_x x} dx \\
&= \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} g_D(n/W_x)\delta(x - n/W_x) e^{-j2\pi m \Delta k_x x} dx \\
&= \sum_{n=0}^{N-1} g_D(n/W_x) e^{-j2\pi m n \Delta k_x / W_x} \\
&= \sum_{n=0}^{N-1} g_D[n] e^{-j2\pi m n / N}
\end{aligned}$$

This is what MATLAB computes when you use fft

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# DFT Basis Functions

DFT: $G[m] = \sum_{n=0}^{N-1} g[n] e^{-j2\pi mn/N}$

Basis Functions are therefore:

$b_m[n] = e^{j2\pi mn/N}$

Are these orthonormal??

Inverse DFT: $g[n] = \frac{1}{N} \sum_{m=0}^{N-1} G[m] e^{j2\pi mn/N}$



# 2D DFT

DFT: $G[r,s] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g[m,n] e^{-j2\pi(rm+sn)/N}$

Basis Functions are therefore:

$b_{r,s}[m,n] = e^{j2\pi(rm+sn)/N}$

Are these orthonormal??

Inverse DFT: $g[m,n] = \frac{1}{N^2} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} G[r,s] e^{j2\pi(rm+sn)/N}$

In general, the number of points along each dimension need not be the same (e.g. $N_1 \neq N_2$). How does this change the expressions?