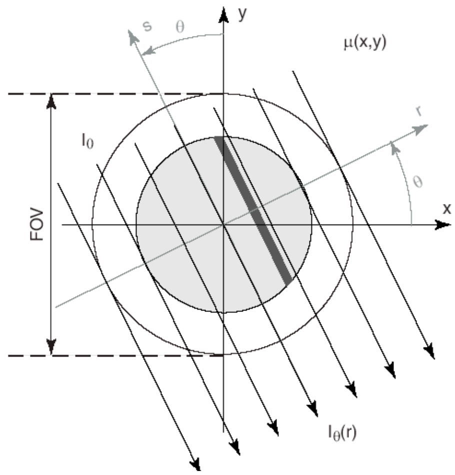
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2006
CT/Fourier Lecture 1

Topics

- Projections and 2D Radon Transform
- Sinograms
- Backprojection
- Projection Slice Theorem
- Fourier Transforms

Projections



$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

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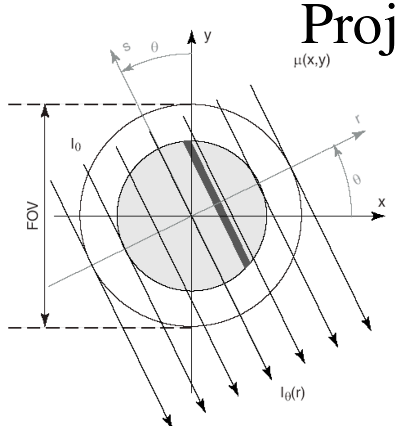
Projections



$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(x,y)\,ds\right)$$
$$= I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)\,ds\right)$$

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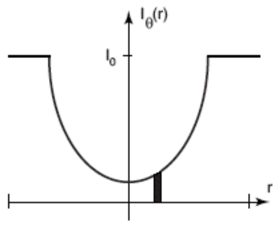
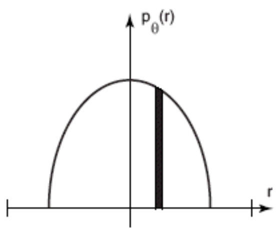
Projections



(b)



(c)

$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

$$p(r,\theta) = -\ln\frac{I_\theta(r)}{I_0}$$

$$= \int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

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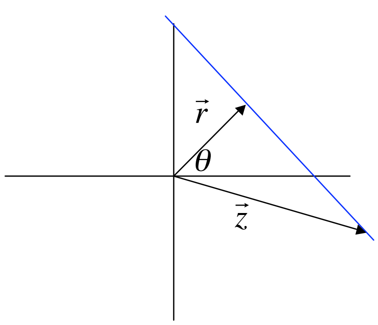
Radon Transform

$$g(r,\theta) = \int_{-\infty}^{\infty} \mu(x(s), y(s))ds$$

$$= \int_{-\infty}^{\infty} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\delta\left(x\cos\theta + y\sin\theta - r\right)dxdy$$



$$\vec{z} \bullet \frac{\vec{r}}{r} = r$$

$$\left(x\hat{x} + y\hat{y}\right) \bullet \left(\cos\theta\hat{x} + \sin\theta\hat{y}\right) = r$$

$$x\cos\theta + y\sin\theta = r$$

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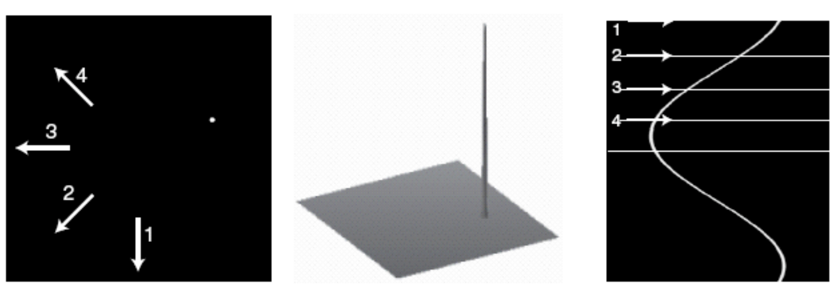
Example

$$f(x,y) = \begin{cases} 1 & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} g(l,\theta = 0) &= \int_{-\infty}^{\infty} f(l,y)\,dy \\ &= \int_{-\sqrt{1-l^2}}^{\sqrt{1-l^2}} dy \\ &= \begin{cases} 2\sqrt{1-l^2} & |l| \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Sinogram



(a) (b) (c)

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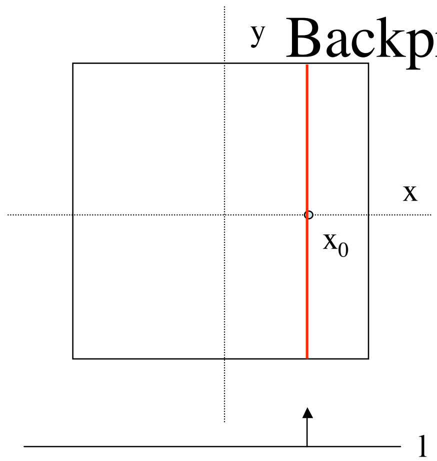
Backprojection

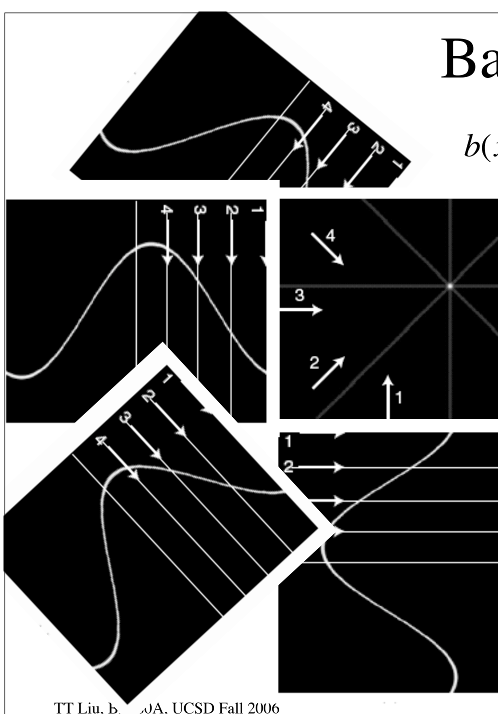


$$b(x_0, y) = p(l, \theta = 0)\Delta\theta$$
$$= p(x_0, 0)\Delta\theta$$

$$b_\theta(x,y) = g(x\cos\theta + y\sin\theta, \theta)\Delta\theta$$
$$b(x,y) = B\{g(l,\theta)\}$$
$$= \int_0^\pi g(x\cos\theta + y\sin\theta, \theta)\,d\theta$$

Suetens 2002

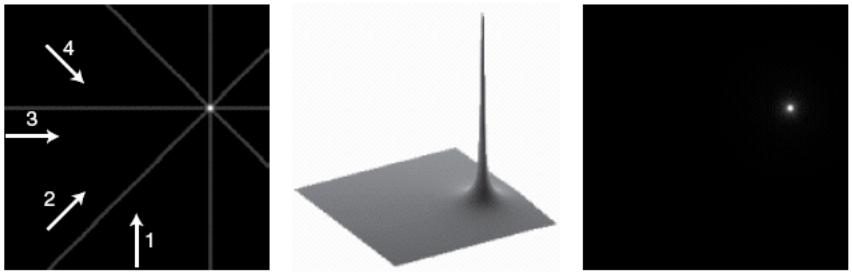
Backprojection



$$b(x,y) = B\{p(l,\theta)\}$$
$$= \int_0^\pi p(x\cos\theta + y\sin\theta, \theta)\,d\theta$$

Suetens 2002

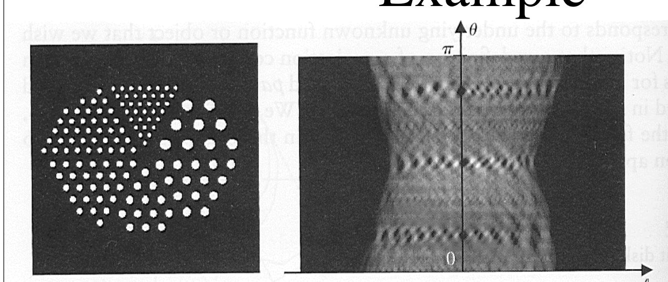
Backprojection



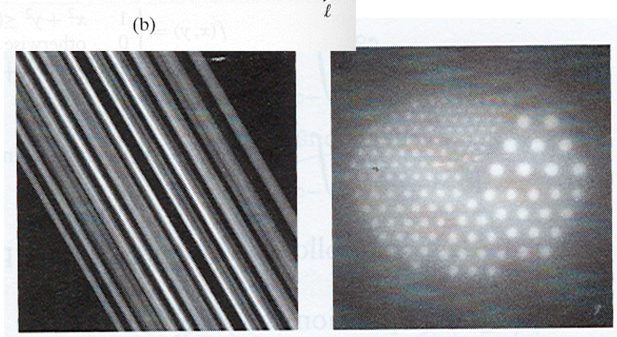
$$b(x,y) = B\{p(l,\theta)\} = \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$

Suetens 2002

Example



(a) (b)



(a) (b)

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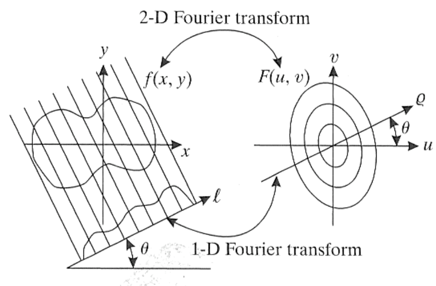
Projection Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$



Prince&Links 2006

The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi k_x x} dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x) e^{j2\pi k_x x} dk_x \qquad \text{Inverse Fourier Transform}$$

2D Fourier Transform

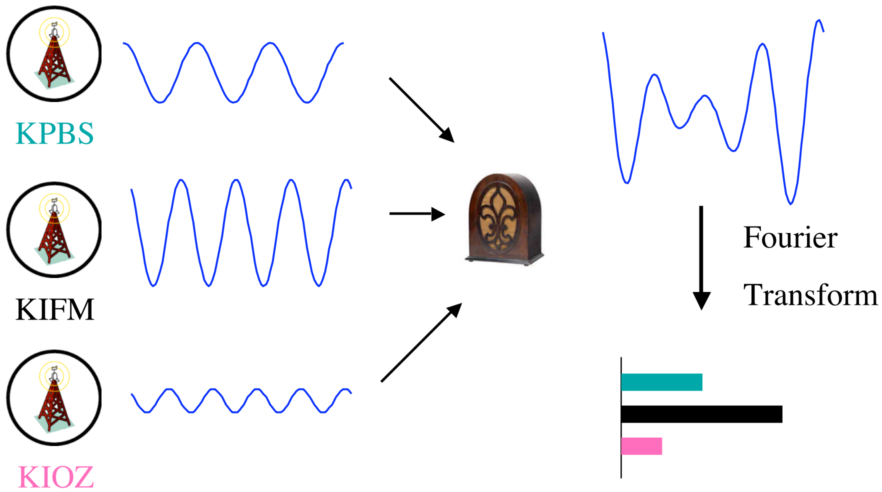
Fourier Transform

$$G(k_x, k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi\left(k_x x + k_y y\right)} dx dy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi\left(k_x x + k_y y\right)} dk_x dk_y$$

1D Fourier Transform



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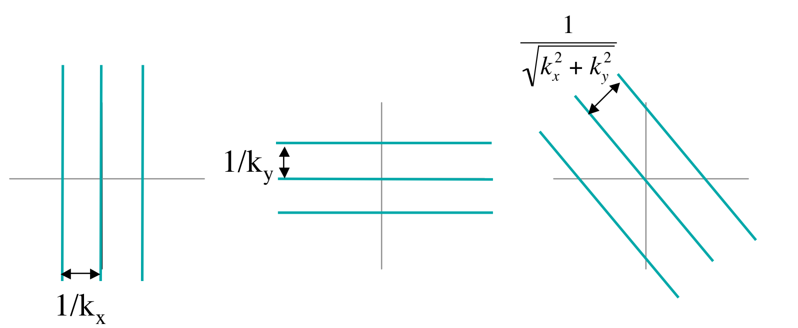
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

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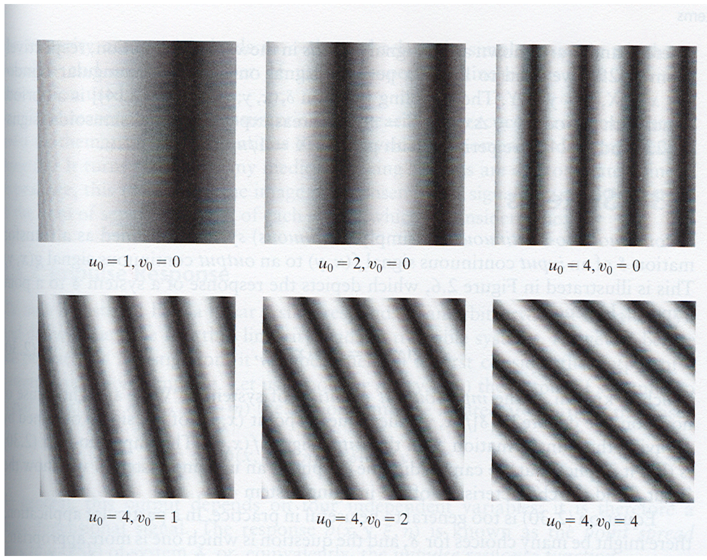
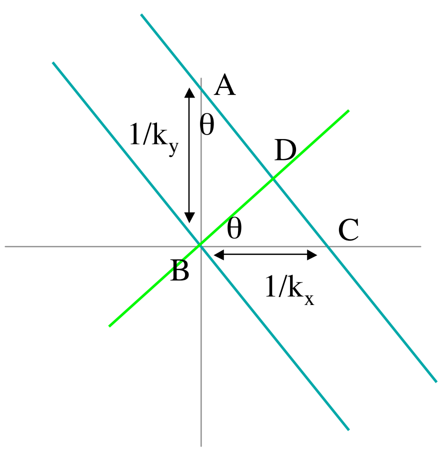


Figure 2.5 from Prince and Link

Plane Waves



$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

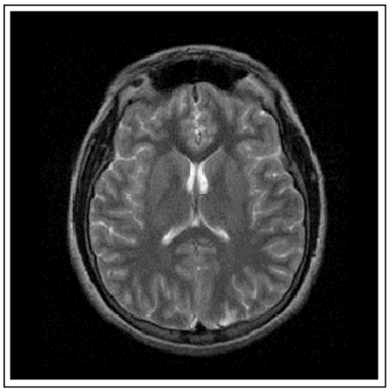
$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

k-space

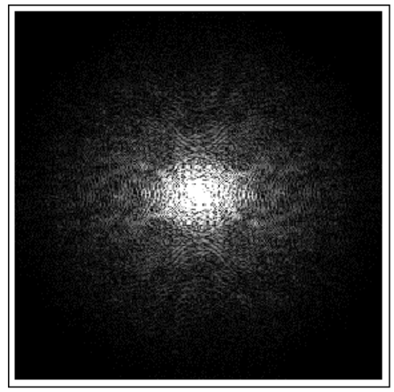
Image space

k-space

y



k_y



x

Fourier Transform

k_x

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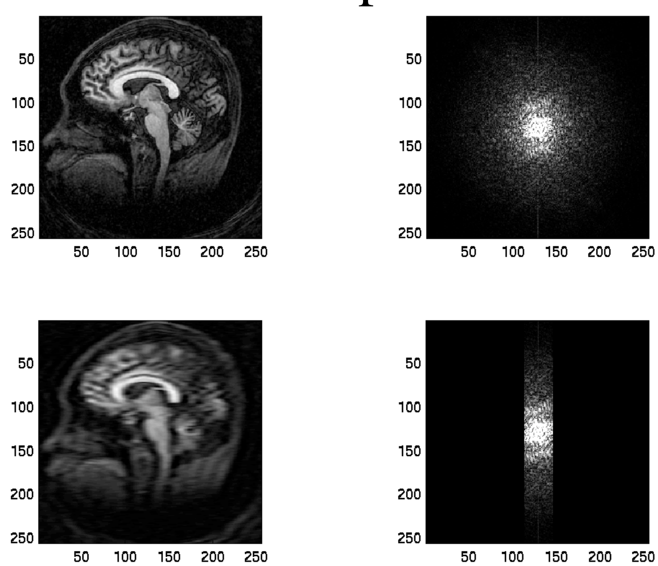
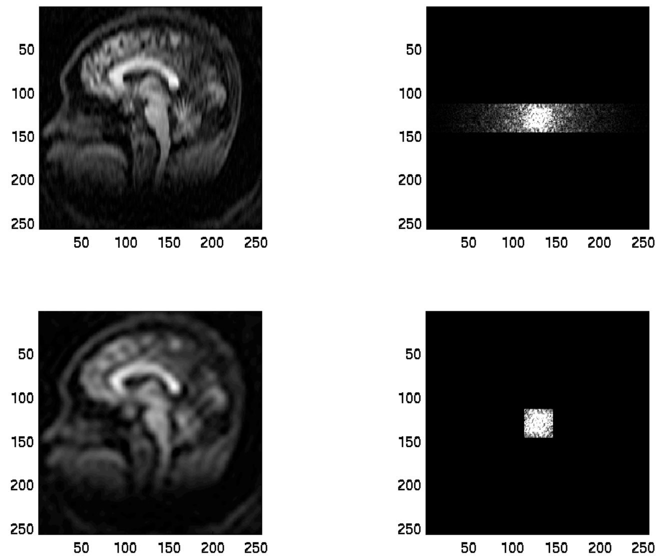
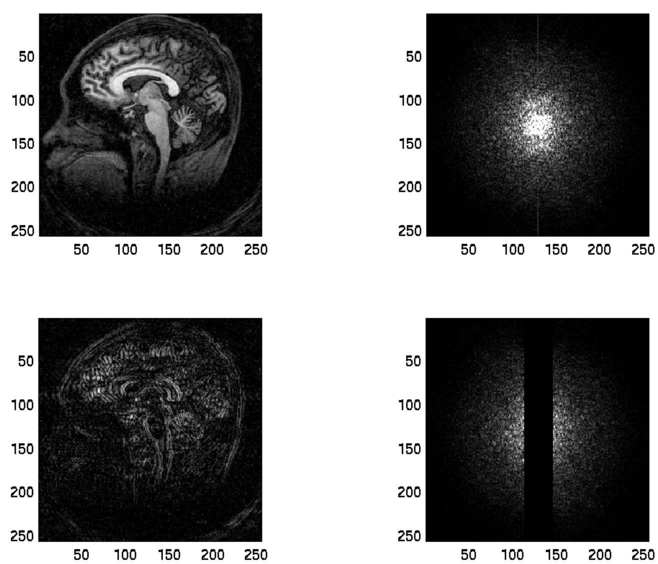
Examples



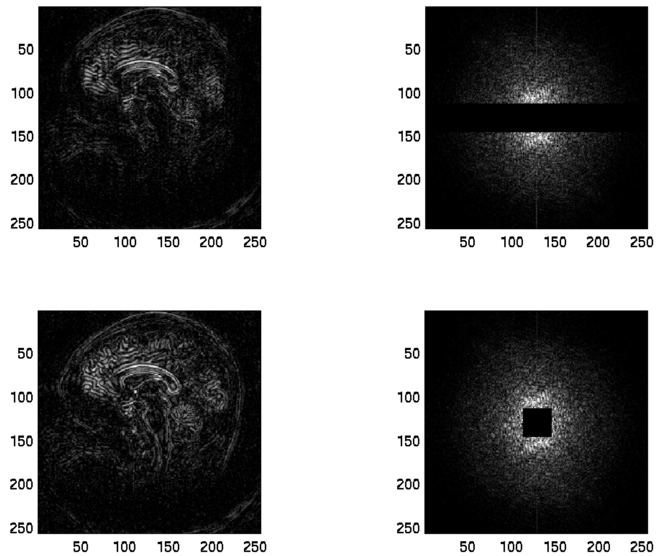
Examples



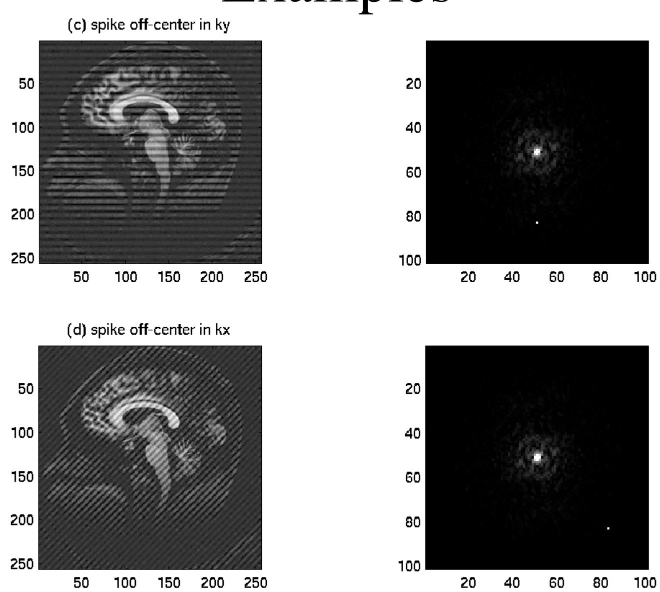
Examples

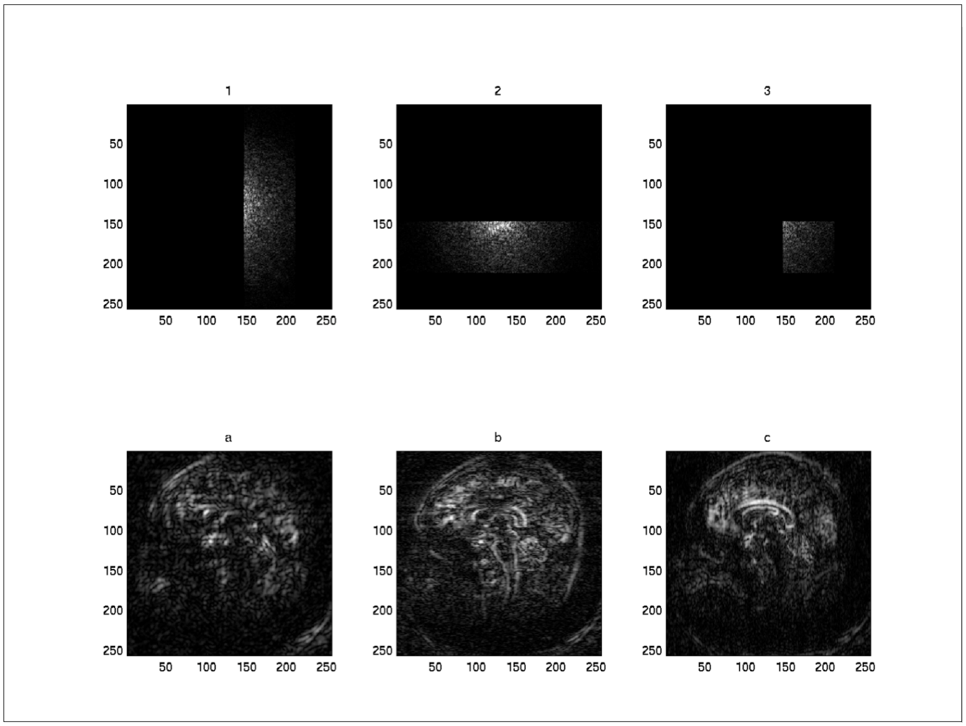


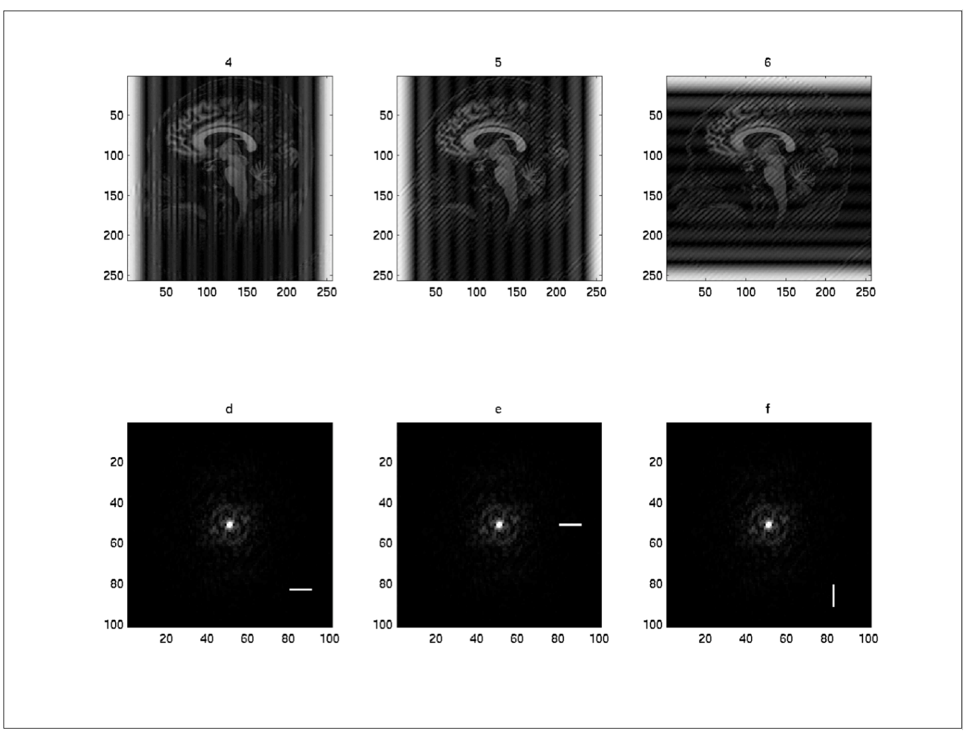
Examples



Examples




1
2
3
a
b
c


4
5
6
d
e
f

Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F\big(\Pi(x)\big) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \mathrm{sin}c(k_x) \end{aligned}$$

Computing Transforms

$$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned} \int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x &= \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx \\ &= \int_{-\infty}^{\infty} g(-x) dx \\ &= G(0) \end{aligned}$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

Computing Transforms

Similarly,

$$F\left\{e^{j2\pi k_0 x}\right\} = \delta(k_x - k_0)$$

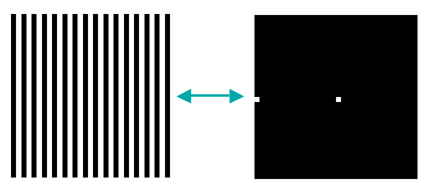
$$F\{\cos 2\pi k_0 x\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$

$$F\{\sin 2\pi k_0 x\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$

Examples

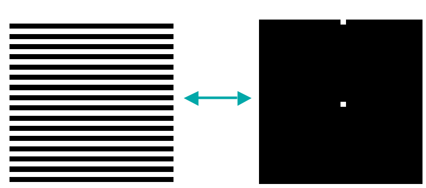
$$g(x,y) = 1 + e^{-j2\pi ax}$$

$$G(k_x,k_y) = \delta\left(k_x,k_y\right) + \delta(k_x + a)\delta(k_y)$$

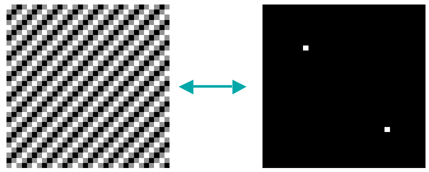


$$g(x,y) = 1 + e^{j2\pi ay}$$

$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x)\delta(k_y - a)$$



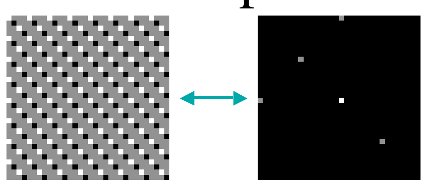
Examples



$$g(x,y) = \cos(2\pi(ax + by))$$

$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

Examples



$$\begin{aligned} G(k_x,k_y) = \delta(k_x,k_y) &+ \\ \delta(k_x + c)\delta(k_y) &+ \\ \delta(k_x)\delta(k_y - d) &+ \\ \frac{1}{2}\delta(k_x - a)\delta(k_y - b) &+ \frac{1}{2}\delta(k_x + a)\delta(k_y + b) \end{aligned}$$

$$g(x,y) = ???$$

Basic Properties

Linearity

$$F\left[ag(x,y)+bh(x,y)\right]=aG(k_x,k_y)+bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right]=\frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

Shift

$$F\left[g(x-a,y-b)\right]=G(k_x,k_y)e^{-j2\pi(k_x a+k_y b)}$$

Modulation

$$F\left[g(x,y)e^{j2\pi(xa+yb)}\right]=G(k_x-a,k_y-b)$$

Linearity

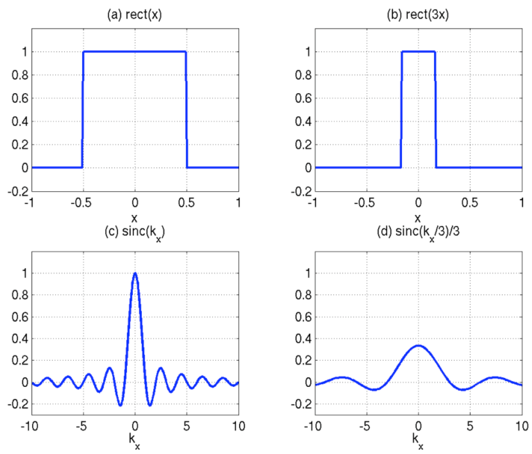
The Fourier Transform is linear.

$$F\left\{ag(x)+bh(x)\right\}=aG(k_x)+bH(k_x)$$

$$F\left[ag(x,y)+bh(x,y)\right]=aG(k_x,k_y)+bH(k_x,k_y)$$

Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right) \qquad F[g(ax,by)] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$



Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$

The Fourier Transform is then separable as well.

$$G(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx \int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy$$
$$= G_X(k_x)G_Y(k_y)$$

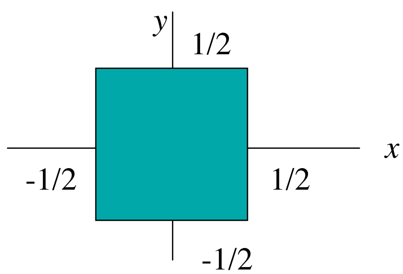
Example

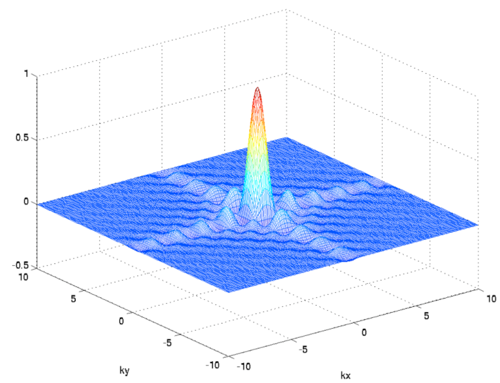
$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Example (sinc/rect)

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$





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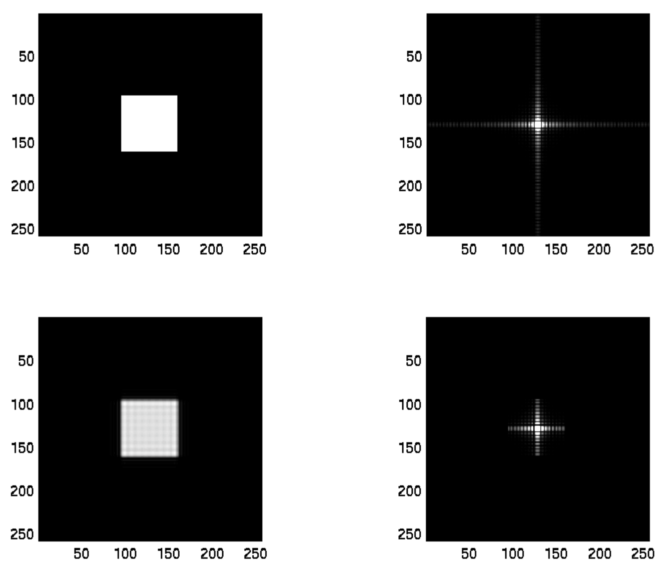
Example (sinc/rect)



Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

Duality

Note the similarity between these two transforms

$$F\left\{e^{j2\pi ax}\right\} = \delta(k_x - a)$$
$$F\left\{\delta(x-a)\right\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\left\{G(x)\right\} = g(-k_x)$$

Application of Duality

$$F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\{\Pi(x)\} = \text{sinc}(k_x)$.

Therefore from duality, $F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x)$

Shift Theorem

$$F\{g(x-a)\} = G(k_x)e^{-j2\pi a k_x}$$

$$F[g(x-a, y-b)] = G(k_x, k_y)e^{-j2\pi(k_x a + k_y b)}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.

Each frequency component is shifted by a. This corresponds to a relative phase shift of

$-2\pi a/(\text{spatial period}) \;=\; -2\pi a k_x$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields $\exp(j2\pi k_x (x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$