Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2006
CT/Fourier Lecture 3

Topics

- Sampling Requirements in CT
- Sampling Theory

CT Sampling Requirements

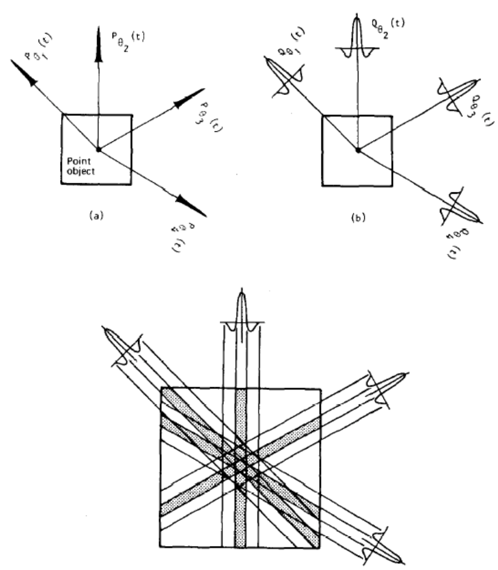
What should the size of the detectors be?

How many detectors do we need?

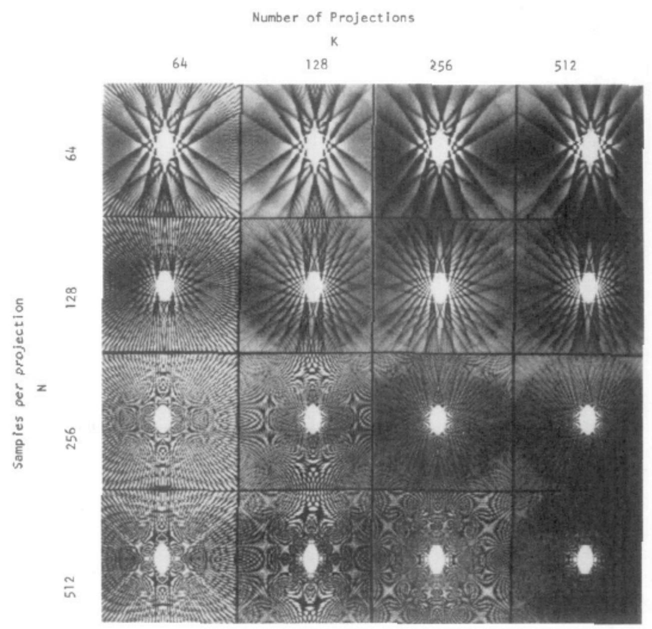
How many views do we need?

Suetens 2002

View Aliasing



Kak and Slaney



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Kak and Slaney

Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

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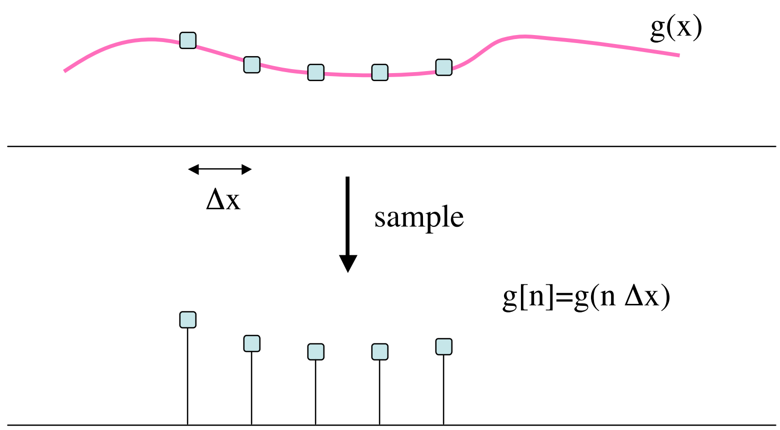
The Process of Sampling



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Questions

How finely do we need to sample?

What happens if we don't sample finely enough?

Can we reconstruct the original signal or image from its samples?

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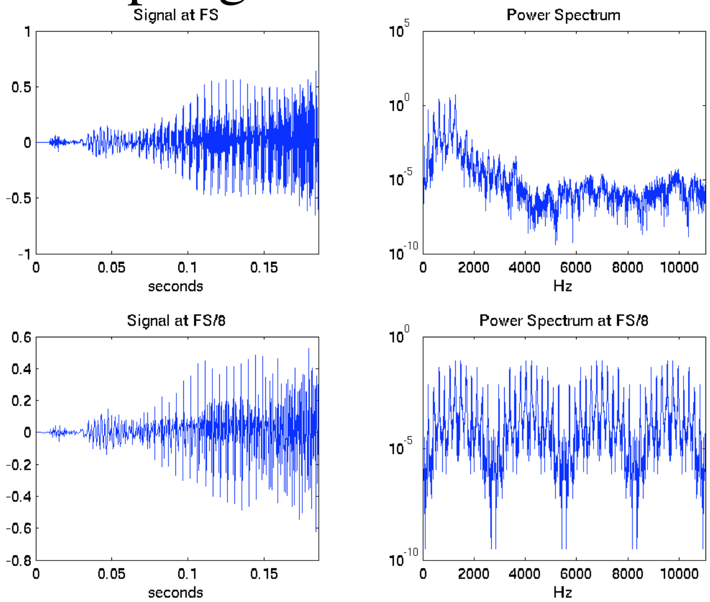
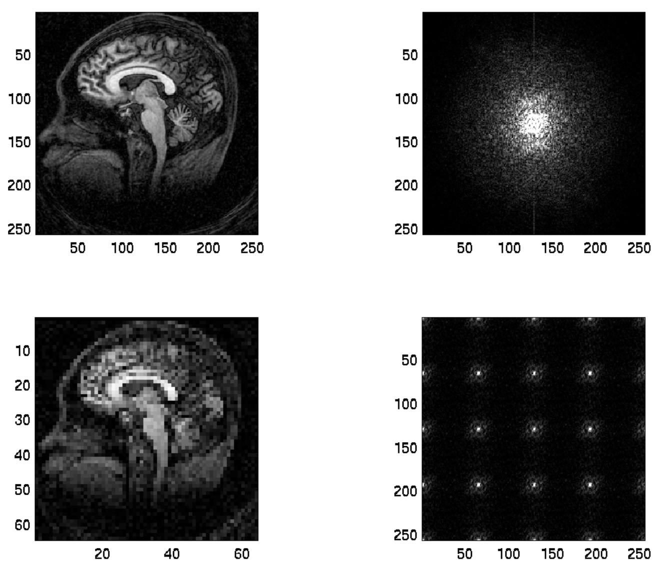
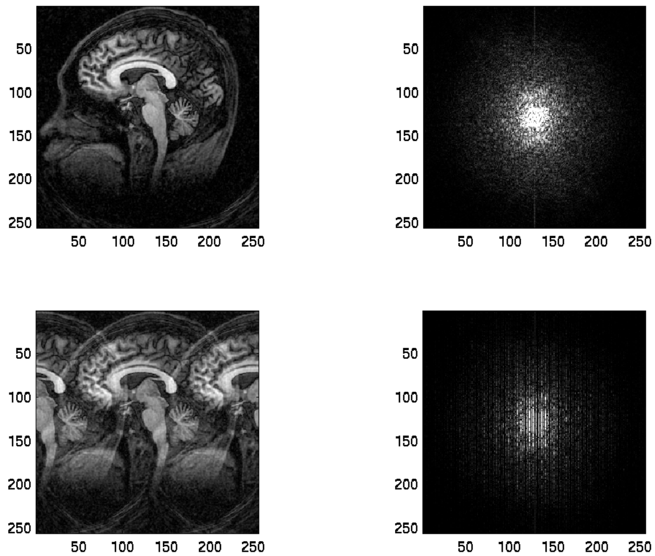
Sampling in the Time Domain



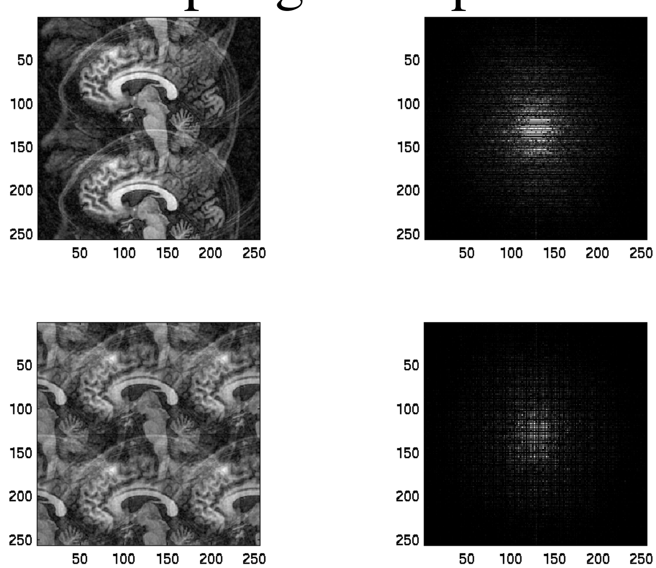
Sampling in Image Space



Sampling in k-space

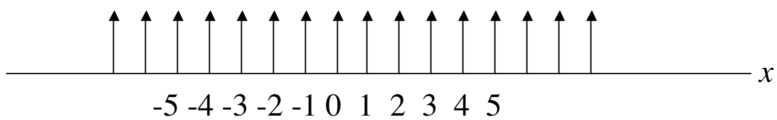


Sampling in k-space



Comb Function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



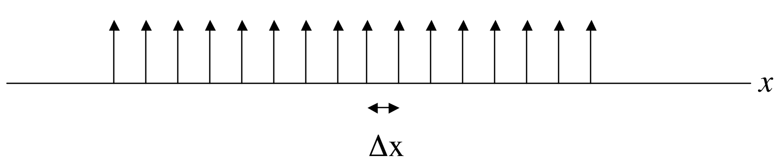
Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$

$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x)\sum_{n=-\infty}^{\infty}\delta(x - n\Delta x)$$

$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty}\delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

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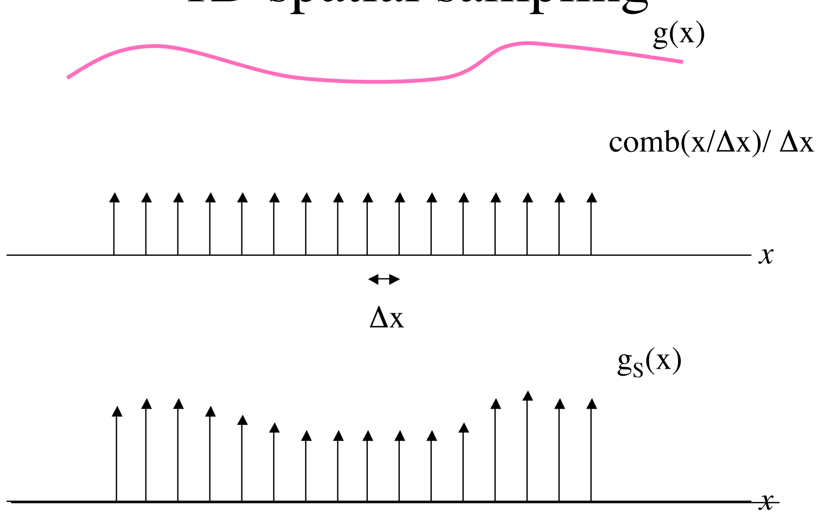
1D spatial sampling



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Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$

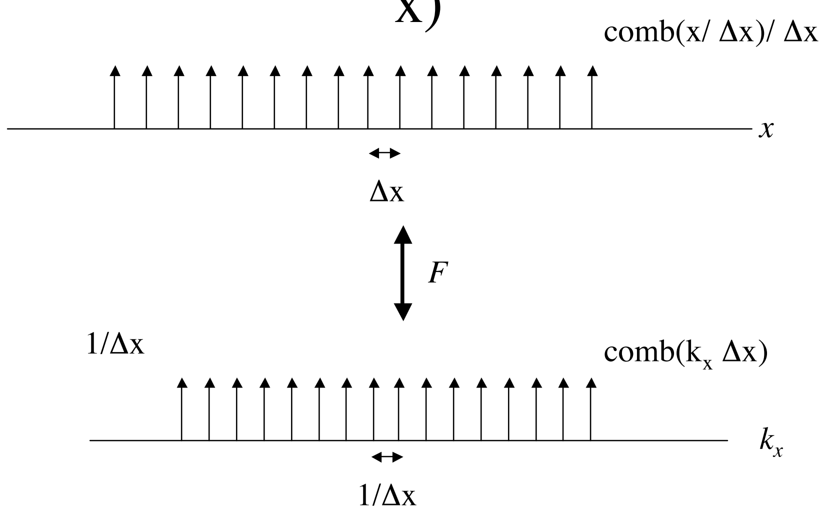
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x} \Delta x comb(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x})$$

Fourier Transform of comb(x/ Δ x)



Fourier Transform of $g_S(x)$

$$F\left[g_S(x)\right] = F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right]$$

$$= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right]$$

$$= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} G\left(k_x - \frac{n}{\Delta x}\right)$$

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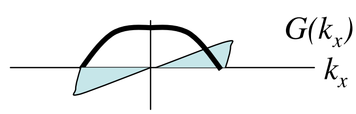
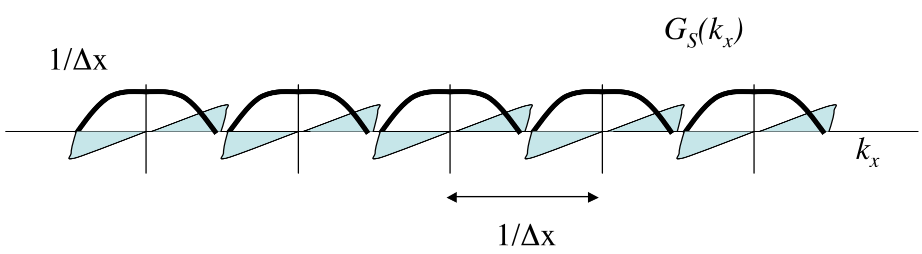
Fourier Transform of $g_S(x)$





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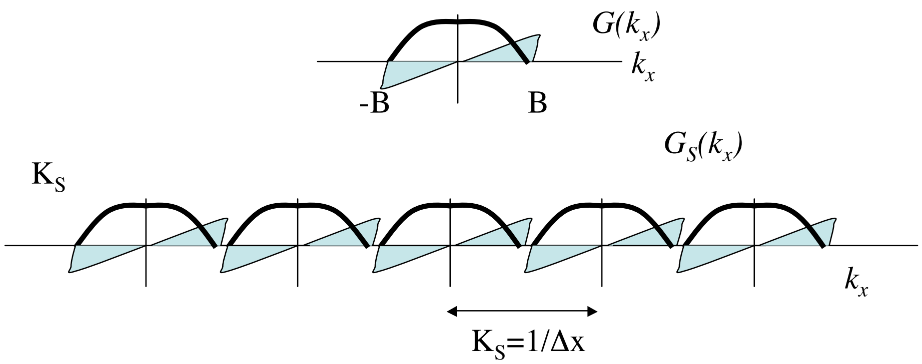
Nyquist Condition



To avoid overlap, we require that $1/\Delta x > 2B$ or
$K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

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Example

Assume that the highest spatial frequency in an object is $B = 2\ cm^{-1}$.
Thus, smallest spatial period is 0.5 cm.

Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

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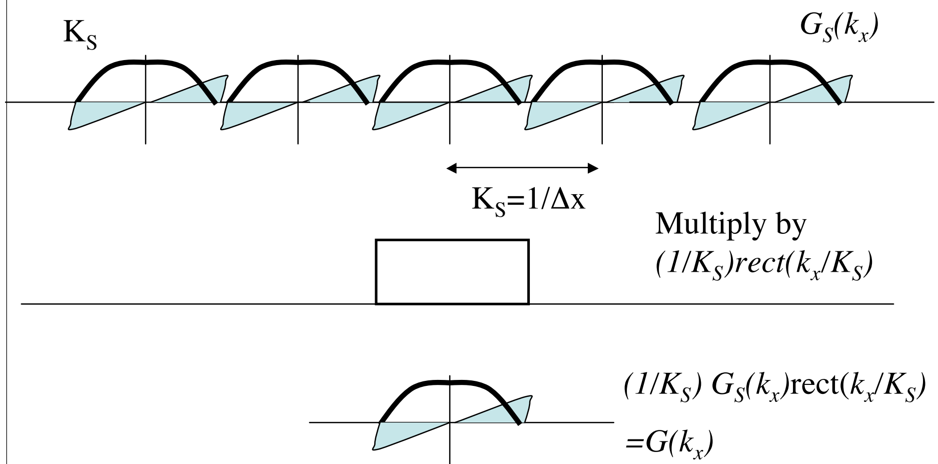
Reconstruction from Samples



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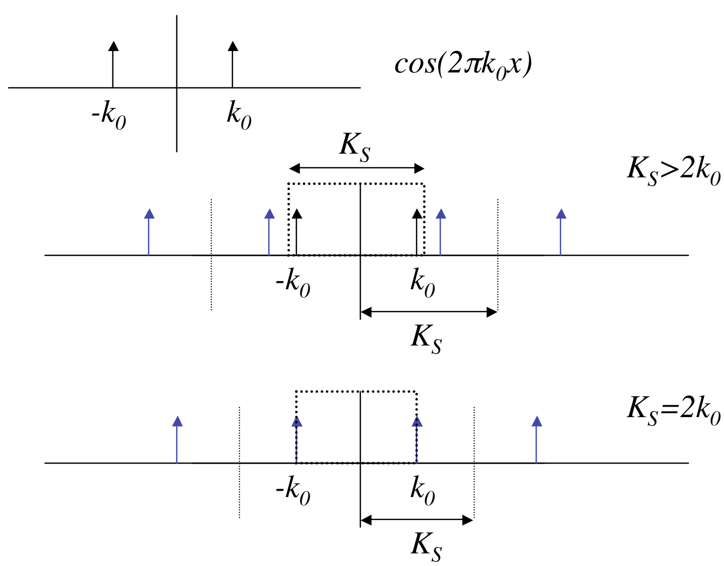
Example Cosine Reconstruction



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Reconstruction from Samples

If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x / K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

$$\begin{aligned}\hat{g}_S(x) &= g_S(x) * \mathrm{sinc}(K_s x) \\ &= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X) \right) * \mathrm{sinc}(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\,\mathrm{sinc}(K_s(x - n\Delta x))\end{aligned}$$

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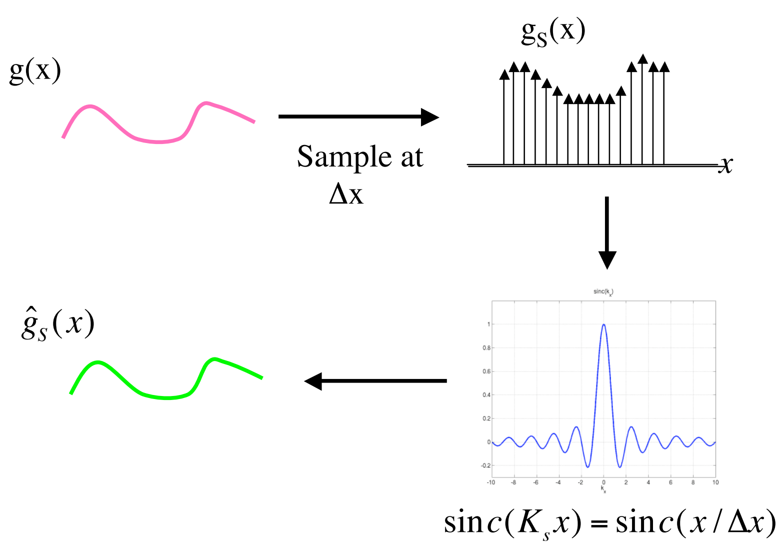
Reconstruction from Samples



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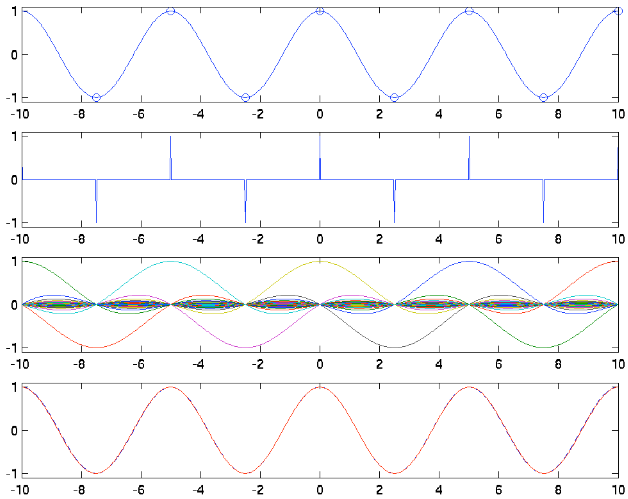
Cosine Example with $K_S=2k_0$



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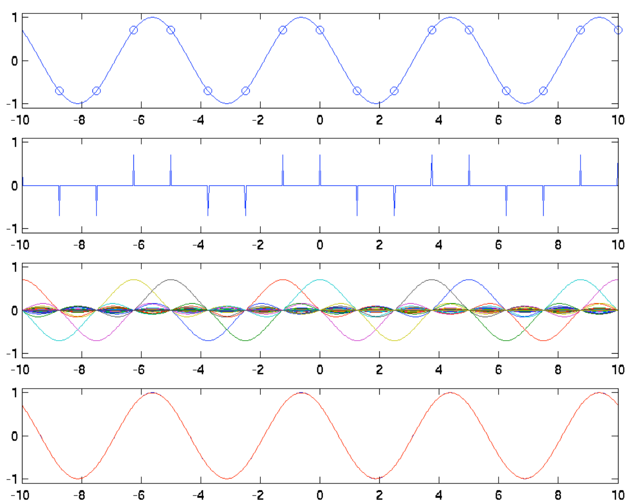
Example with $K_s=4k_0$



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Example with $K_s = 8k_0$