Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2006
X-Rays/CT Lecture 4

Topics

- Review Signal Expansions, Impulse Response and Linearity
- Superposition
- Space Invariance
- Convolution
- Start Computed Tomography

Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$

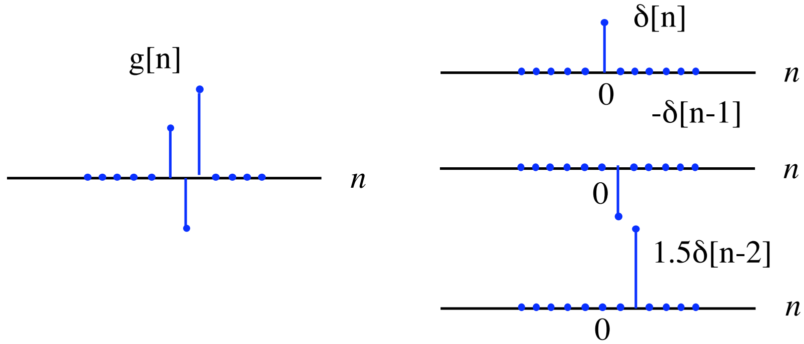


Image Decomposition

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$g[m,n] = a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1]$$

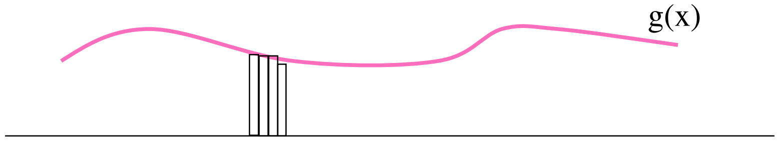
$$= \sum_{k=0}^{1} \sum_{l=0}^{1} g[k,l]\delta[m-k,n-l]$$

Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation

$$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$$



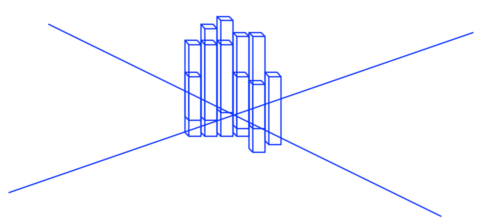
Representation of 2D Function

Similarly, we can write a 2D function as

$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$

To gain intuition, consider the approximation

$$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$



Impulse Response

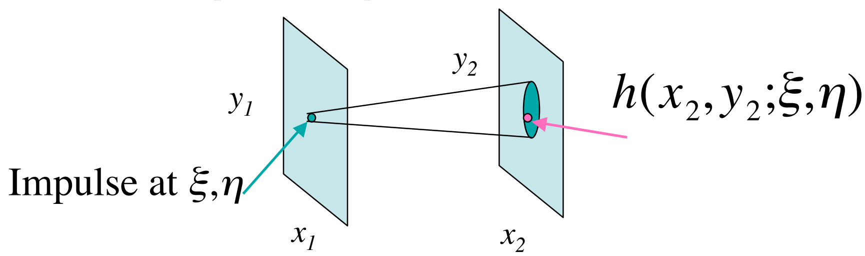
The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$$h(x_2;\xi) = L\left[\delta(x_1-\xi)\right]$$
$$h[m',k] = L\left[\delta[m-k]\right]$$

1D Impulse Response

$$h(x_2,y_2;\xi,\eta) = L\left[\delta(x_1-\xi,y_1-\eta)\right]$$
$$h[m',n';k,l] = L\left[\delta[m-k,n-l]\right]$$

2D Impulse Response



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$

Superposition

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$
$$h[m',k] = L[\delta[m-k]]$$

$$\begin{aligned}
y[m'] &= L\big[g[m]\big] \\
&= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\
&= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\
&= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\
&= g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \\
&= \sum_{k=0}^{2} g[k]h[m',k]
\end{aligned}$$

Superposition Integral

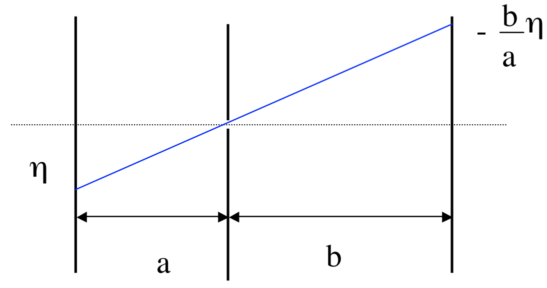
What is the response to an arbitrary function $g(x_1, y_1)$?

Write $g(x_1, y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi, y_1-\eta)d\xi d\eta.$

The response is given by

$$\begin{aligned}
I(x_2, y_2) &= L\big[g_1(x_1, y_1)\big] \\
&= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi, y_1-\eta)d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L\big[\delta(x_1-\xi, y_1-\eta)\big]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2, y_2; \xi,\eta)d\xi d\eta
\end{aligned}$$

Pinhole Magnification Example

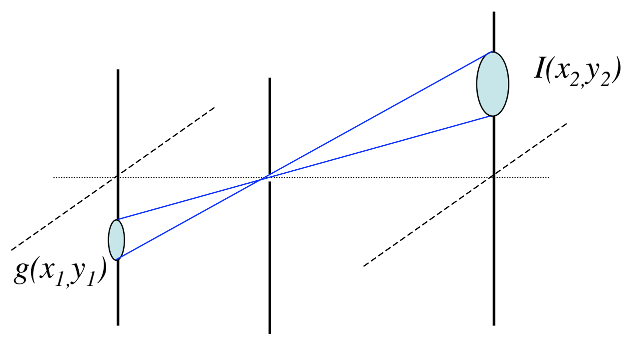


In this example, an impulse at (ξ,η) will yield an impulse at $(m\xi,m\eta)$ where $m=-b/a$.

Thus, $h(x_2,y_2;\xi,\eta)=L\left[\delta(x_1-\xi,y_1-\eta)\right]=\delta(x_2-m\xi,y_2-m\eta)$.

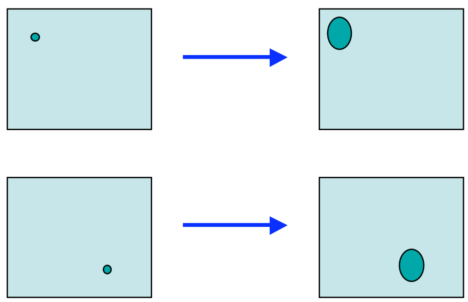
Pinhole Magnification Example

$$I(x_2,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x_2-m\xi,y_2-m\eta)d\xi d\eta$$

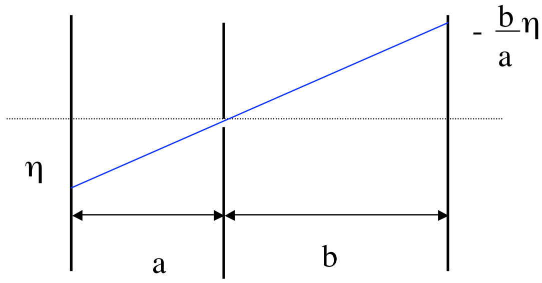


Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h\left(x_2-\xi, y_2-\eta\right)$



Pinhole Magnification Example

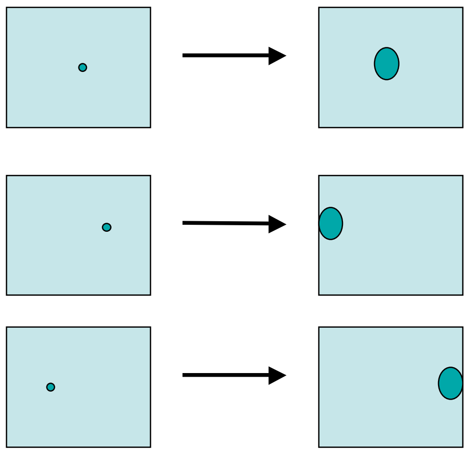


$h\left(x_2,y_2;\xi,\eta\right) = C\delta(x_2 - m\xi, y_2 - m\eta)$.

Is this system space invariant?

Pinhole Magnification Example

____, the pinhole system ____ space invariant.



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Convolution

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]] = h[m-k]$$

$$\begin{aligned}
y[m'] &= L\big[g[m]\big] \\
&= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\
&= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\
&= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\
&= g[0]h[m'-0] + g[1]h[m'-1] + g[2]h[m'-2] \\
&= \sum_{k=0}^{2} g[k]h[m'-k]
\end{aligned}$$

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1D Convolution

$$
\begin{aligned}
I(x) &= \int_{-\infty}^{\infty} g(\xi) h(x;\xi) d\xi \\
&= \int_{-\infty}^{\infty} g(\xi) h(x-\xi) d\xi \\
&= g(x) * h(x)
\end{aligned}
$$

Useful fact:

$$
\begin{aligned}
g(x) * \delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi) \delta(x-\Delta-\xi) d\xi \\
&= g(x-\Delta)
\end{aligned}
$$

2D Convolution

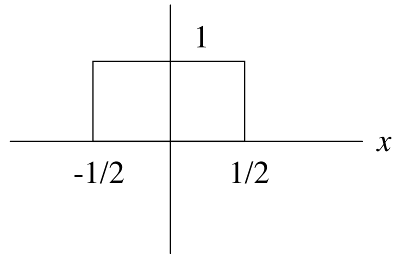
For a space invariant linear system, the superposition integral becomes a convolution integral.

$$
\begin{aligned}
I(x_2, y_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) h(x_2, y_2;\xi,\eta) d\xi d\eta \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) h(x_2-\xi, y_2-\eta) d\xi d\eta \\
&= g(x_2, y_2) ** h(x_2, y_2)
\end{aligned}
$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)$.

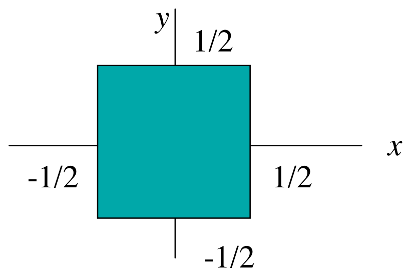
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$



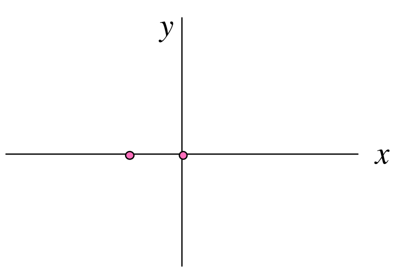
Also called rect(x)

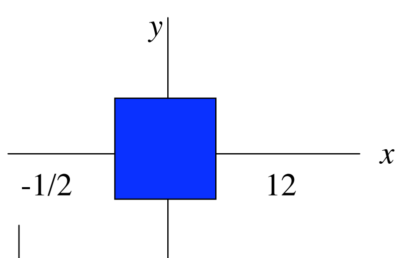
$$\Pi(x,y) = \Pi(x)\Pi(y)$$



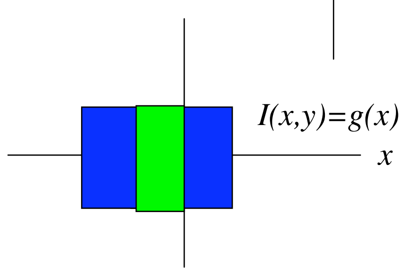
2D Convolution Example

g(x)= $\delta(x+1/2,y) + \delta(x,y)$ *h(x)=rect(x,y)*





*I(x,y)=g(x)**h(x,y)*



2D Convolution Example

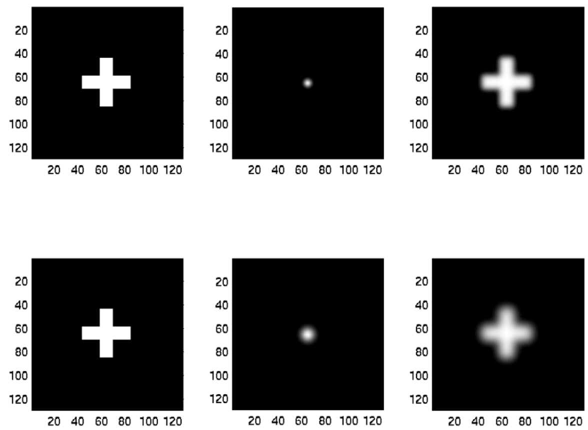


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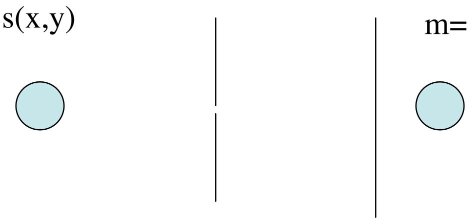
Image of a point object

s(x,y)

$$I_d(x,y) = \lim_{m \to 0} ks(x/m, y/m)$$
$$= \delta(x,y)$$

s(x,y)

m=1



$$I_d(x,y) = s(x,y)$$

In general, $I_d(x,y) = \frac{1}{m^2} s(x/m, y/m)$

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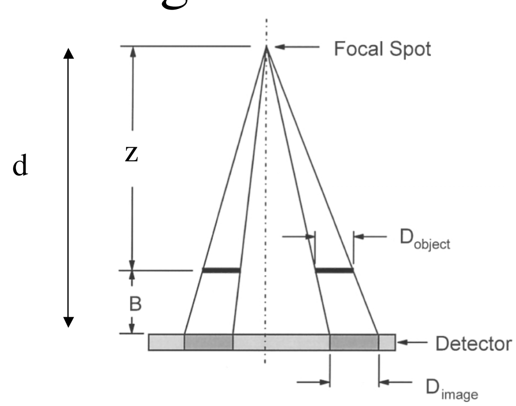
Pinhole Magnification Example

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)\delta(x_2 - m\xi, y_2 - m\eta)d\xi d\eta$$

after substituting $\xi' = m\xi$ and $\eta' = m\eta$, we obtain

$$= \frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi'/m,\eta'/m)\delta(x_2 - \xi', y_2 - \eta')d\xi' d\eta'$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m) ** \delta(x_2,y_2)$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m)$$

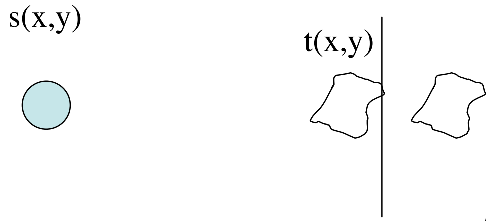
Magnification of Object



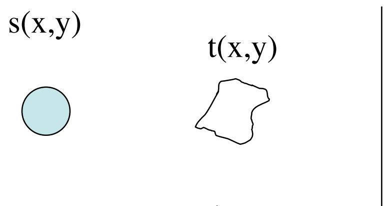
$$M(z) = \frac{d}{z}$$
$$= \frac{\text{Source to Image Distance (SID)}}{\text{Source to Object Distance (SOD)}}$$

Bushberg et al 2001

Image of arbitrary object



$$\lim_{m\to 0} I_d(x,y) = t(x,y)$$



m=1

$$I_d(x,y) = ???$$

$$I_d(x,y) = \frac{1}{m^2} s(x/m, y/m) ** t(x/M, y/M)$$

Note: we are ignoring obliquity, etc.

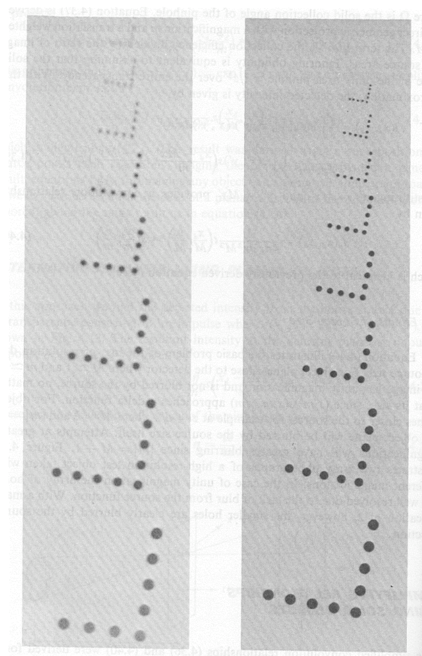
X-Ray Image Equation

$$\begin{aligned} I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta) h(x_2,y_2;\xi,\eta) d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta) t\left(\frac{x_2 - m\xi}{M}, \frac{y_2 - m\eta}{M}\right) d\xi d\eta \end{aligned}$$

after substituting $\xi' = m\xi$ and $\eta' = m\eta$, we obtain

$$\begin{aligned} &= \frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi'/m, \eta'/m) t\left(\frac{x_2 - \xi'}{M}, \frac{y_2 - \eta'}{M}\right) d\xi' d\eta' \\ &= \frac{1}{m^2} s(x_2/m, y_2/m) ** t\left(x_2/M, y_2/M\right) \end{aligned}$$

Note: we have ignored obliquity factors etc.



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Macovski 1983

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.
3. Dirac delta functions are generalized functions.

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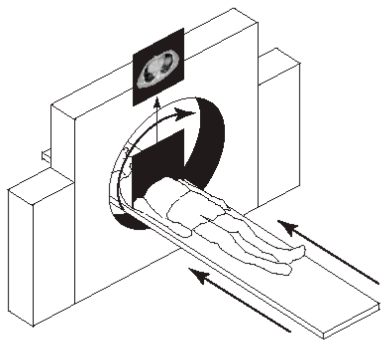
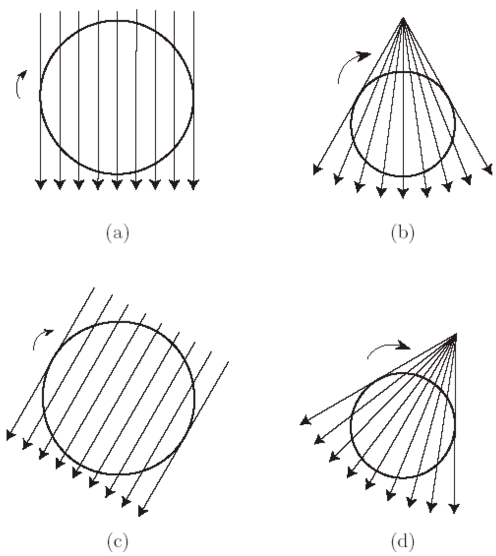
Computed Tomography



Suetens 2002

Computed Tomography

Parallel Beam



Fan Beam

Suetens 2002

Computed Tomography

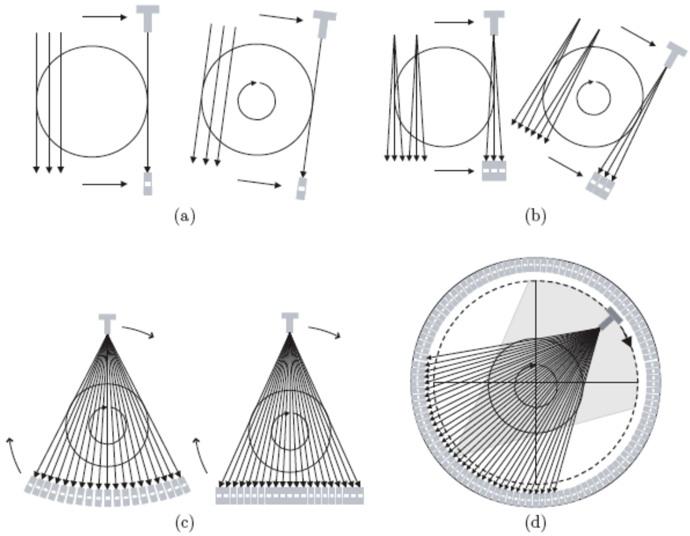


Figure 5.19: *Subsequent scanner generations: (a) first generation, (b) second generation, (c) third generation and (d) fourth generation CT scanner.*

Suetens 2002

Computed Tomography

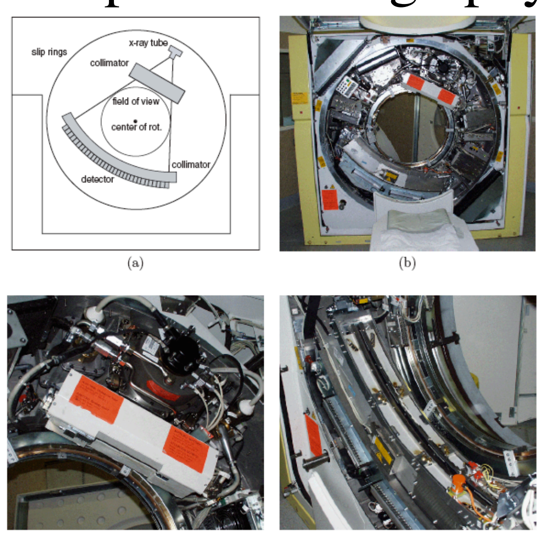


Figure 5.20: *(a-b) The basic internal geometry of a third generation spiral CT scanner. (c) X-ray tube with adjustable collimating split. (d) Detector array with post-patient collimator.*

Suetens 2002

Computed Tomography

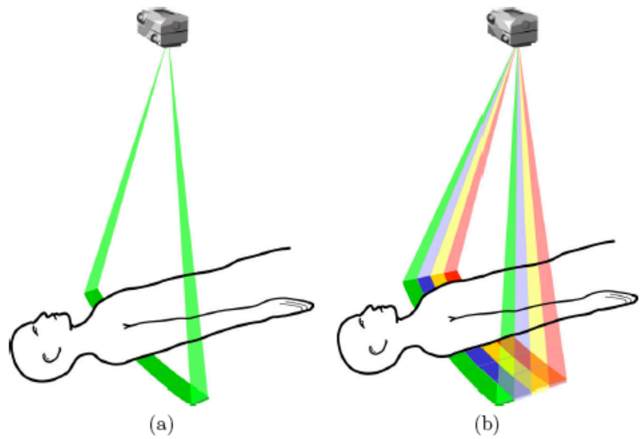


Figure 5.22: *(a) Single-slice CT versus (b) multi-slice CT: a multi-slice CT scanner can acquire four slices simultaneously by using four adjacent detector arrays (Reprinted with permission of RSNA).*

Suetens 2002

CT Line Integral

$$I_d = \int_0^{E_{\max}} S_0(E) E \exp\left(-\int_0^d \mu(s;E')ds\right)dE$$

Monoenergetic Approximation

$$I_d = I_0 \exp\left(-\int_0^d \mu(s;\overline{E})ds\right)$$

$$g_d = -\log\left(\frac{I_d}{I_0}\right)$$

$$= \int_0^d \mu(s;\overline{E})ds$$

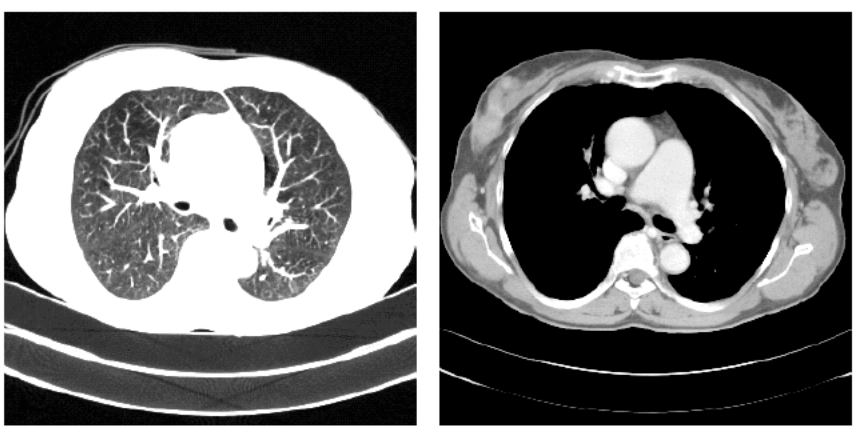
CT Number

$$\text{CT\_number} = \frac{\mu - \mu_{water}}{\mu_{water}} \times 1000$$

Measured in Hounsfield Units (HU)

Air: -1000 HU
Soft Tissue: -100 to 60 HU
Cortical Bones: 250 to 1000 HU
Metal and Contrast Agents: > 2000 HU

CT Display



(a) (b)

Figure 5.4: *CT-image of the chest with different window/level settings:(a) for the lungs (window 1500 and level -500) and (b) for the soft tissues (window 350 and level 50).*

Suetens 2002

Direct Inverse Approach

$$\begin{array}{cc|c} \mu_1 & \mu_2 & p_1 \\ \mu_3 & \mu_4 & p_2 \\ \hline p_3 & p_4 & \end{array}$$

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_4 = \mu_2 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns.
Are these the correct equations to use?

Direct Inverse Approach

$$\begin{array}{cc|c} \mu_1 & \mu_2 & p_1 \\ \mu_3 & \mu_4 & p_2 \\ \hline p_3 & p_4 & \end{array}$$

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_4 = \mu_2 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns.
Are these the correct equations to use?

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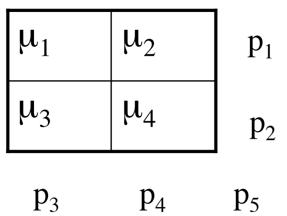
Direct Inverse Approach

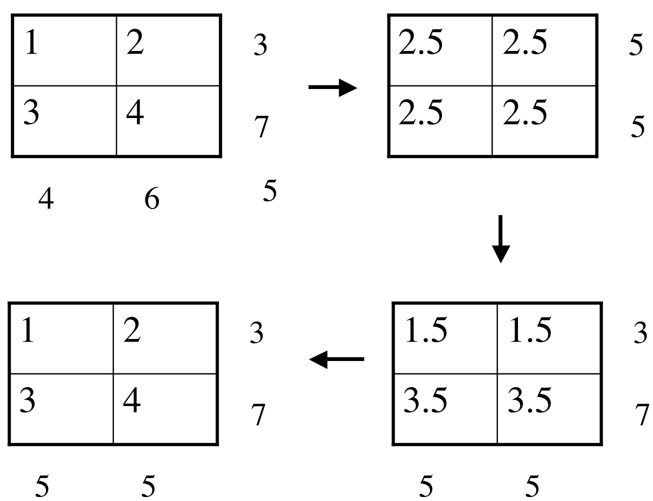


$$p_1 = \mu_1 + \mu_2$$
$$p_2 = \mu_3 + \mu_4$$
$$p_3 = \mu_1 + \mu_3$$
$$p_5 = \mu_1 + \mu_4$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns. These are linearly independent now.
In general for a NxN image, N^2 unknowns, N^2 equations.
This requires the inversion of a N^2xN^2 matrix
For a high-resolution 512x512 image, N^2=262144 equations.
Requires inversion of a 262144x262144 matrix!
Inversion process sensitive to measurement errors.

Iterative Inverse Approach
Algebraic Reconstruction Technique (ART)



Backprojection

0	0	0	0
0	3	0	3
0	0	0	0
3	0 3 0		3

0	0	0
1	1	1
0	0	0

→

1	0	0
1	2	1
0	0	1

→

1	1	0
1	3	1
0	1	1

→

1	1	1
1	4	1
1	1	1

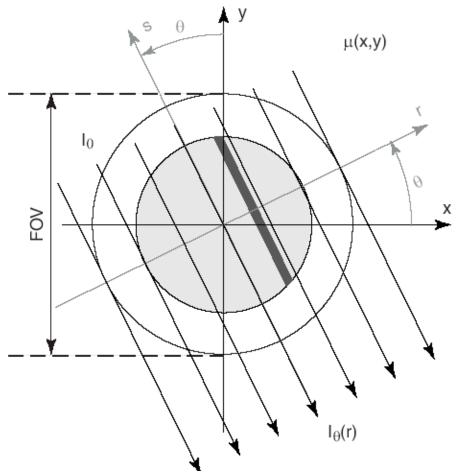
Suetens 2002

In-Class Exercise

μ_1	μ_2	5.7
μ_3	μ_4	11.3
8.2	8.8	10.1

Suetens 2002

Projections



$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

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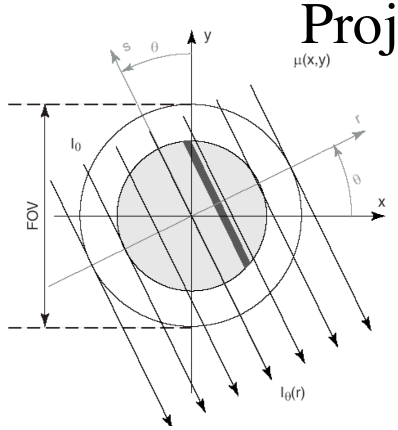
Projections



$$I_\theta(r) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(x,y)ds\right)$$
$$= I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

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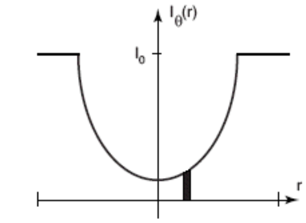
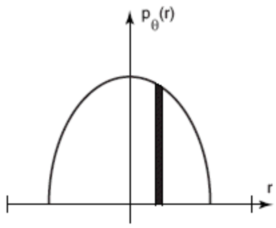
Projections



(b)

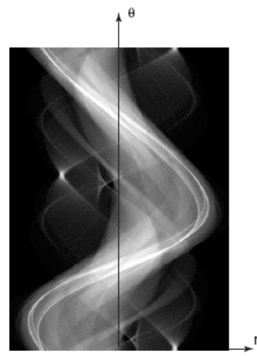


(c)

$$I_\theta(r) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

$$p_\theta(r) = -\ln\frac{I_\theta(r)}{I_0}$$

$$= \int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$



Sinogram

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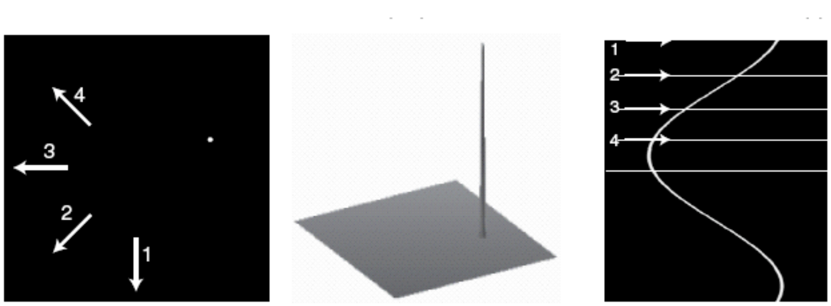
Sinogram



(a) (b) (c)

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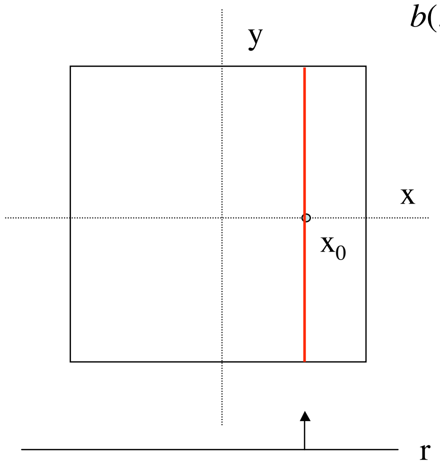
Backprojection



$$b(x,y) = B\{p(r,\theta)\}$$
$$= \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)d\theta$$

$$b(x_0,y) = p(r,\theta = 0)\Delta\theta$$
$$= p(x_0)\Delta\theta$$

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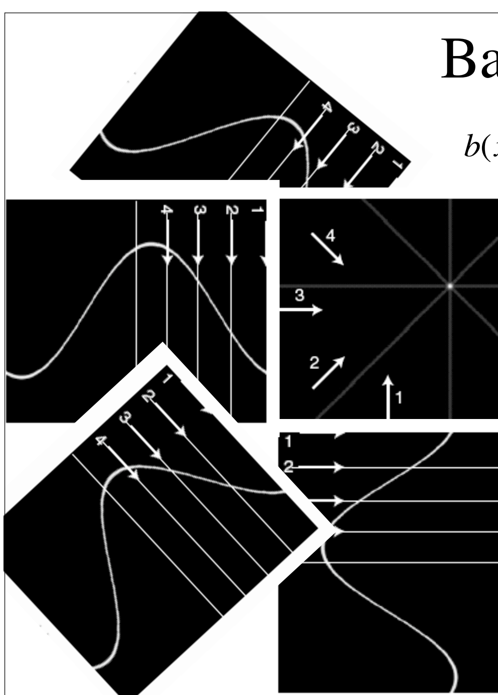
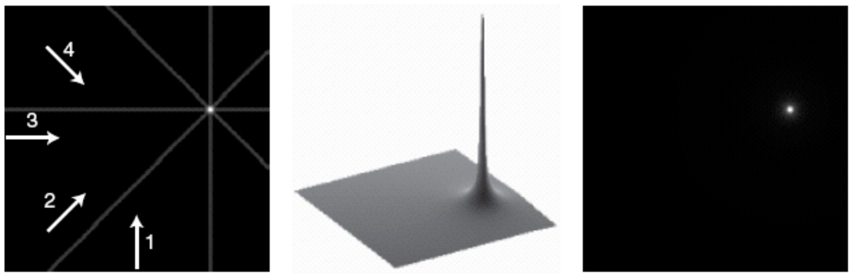
Backprojection



$$b(x,y) = B\{p(r,\theta)\}$$
$$= \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)d\theta$$

Suetens 2002

Backprojection



$$b(x,y) = B\left\{p\left(r,\theta\right)\right\} = \int_0^{\pi} p(x\cos\theta + y\sin\theta,\theta)d\theta$$

Suetens 2002