Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2008
CT/Fourier Lecture 3

Topics

- Modulation
- Modulation Transfer Function
- Convolution/Multiplication
- Revisit Projection-Slice Theorem
- Filtered Backprojection

Modulation

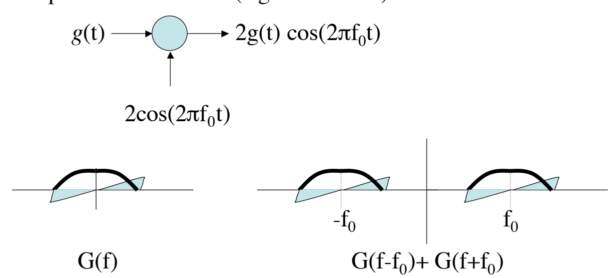
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

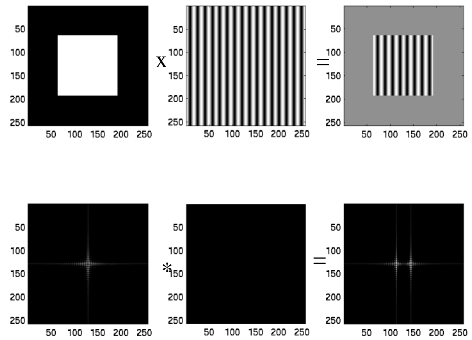
$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

Example

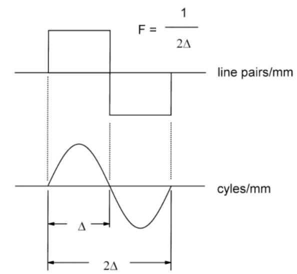
Amplitude Modulation (e.g. AM Radio)

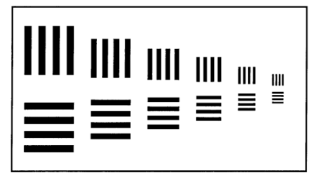


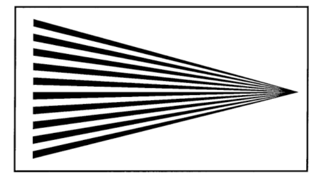
Modulation Example



Bushberg et al 2001



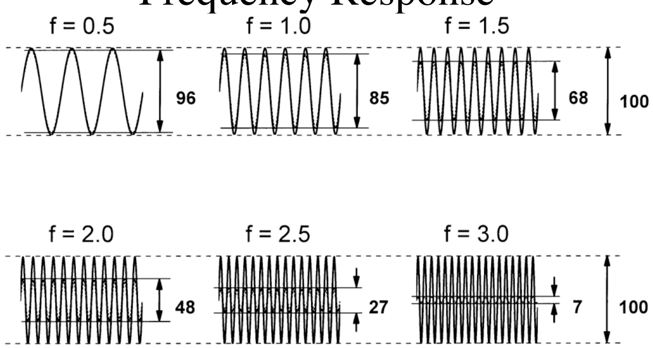




Line Pair Test Phantom

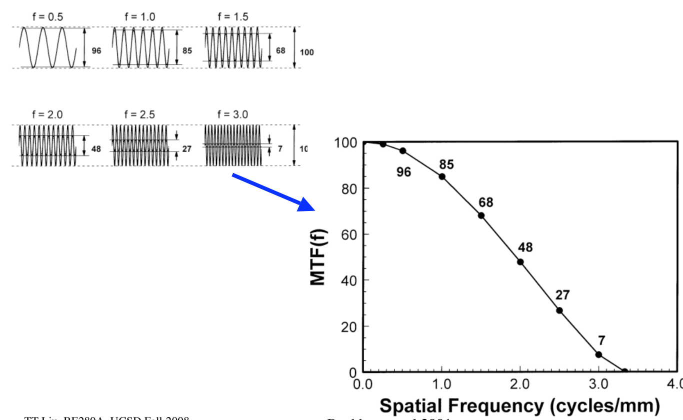
Section of a Star Pattern

Modulation Transfer Function (MTF) or Frequency Response



Bushberg et al 2001

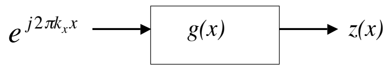
Modulation Transfer Function



Bushberg et al 2001

Eigenfunctions

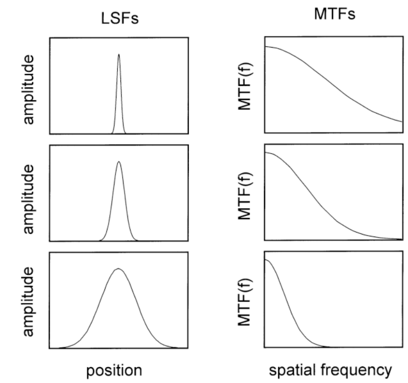
The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.



$$z(x) = g(x) * e^{j2\pi k_x x}$$
$$= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du$$
$$= G(k_x) e^{j2\pi k_x x}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

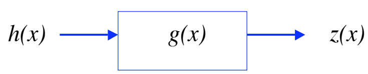
MTF = Fourier Transform of PSF



Bushberg et al 2001

Convolution/Multiplication

Now consider an arbitrary input *h(x)*.



Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution/Modulation Theorem

$$F\{g(x) * h(x)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du$$
$$= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du$$
$$= G(k_x) H(k_x)$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

2D Convolution/Multiplication

Convolution

$$F\left[g(x,y) ** h(x,y)\right] = G(k_x,k_y)H(k_x,k_y)$$

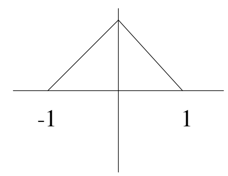
Multiplication

$$F\left[g(x,y)h(x,y)\right] = G(k_x,k_y) ** H(k_x,k_y)$$

Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

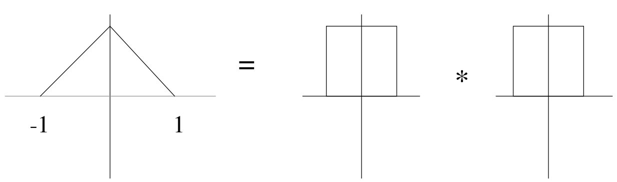
$$F(\Lambda(x)) = \int_{-1}^{1}\left(1-|x|\right)e^{-j2\pi k_x x}dx = ??$$



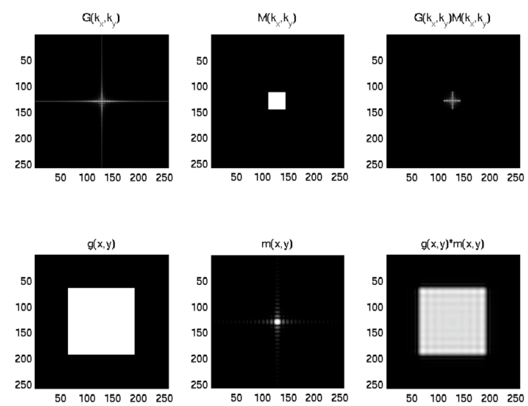
Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$

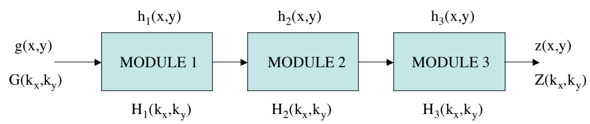
$$F(\Lambda(x)) = \text{sinc}^2(k_x)$$



Convolution Example



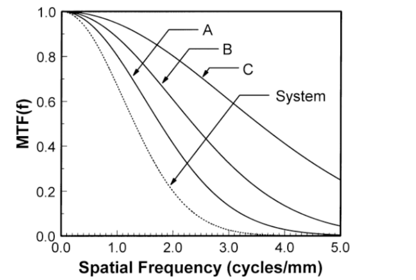
Response of an Imaging System



$z(x,y)=g(x,y)**h_1(x,y)**h_2(x,y)**h_3(x,y)$

$Z(k_x,k_y)=G(k_x,k_y)\ H_1(k_x,k_y)\ H_2(k_x,k_y)\ H_3(k_x,k_y)$

System MTF = Product of MTFs of Components



Bushberg et al 2001

Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots FWHM_N^2}$$

Example

$$FWHM_1 = 1mm$$

$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$

Figure 1:

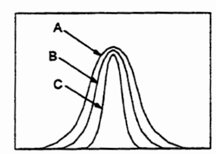
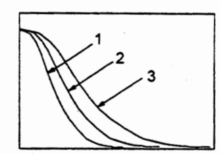


Figure 2:



8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSF), which LSF will yield the best spatial resolution?

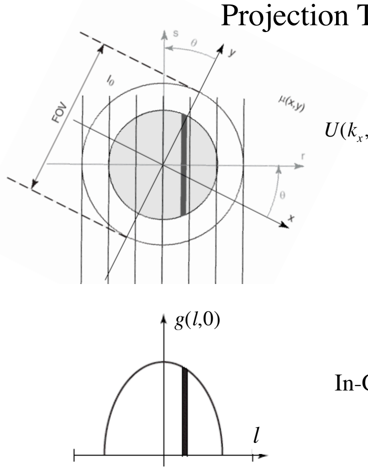
10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

    A. MTF number 1
    B. MTF number 2
    C. MTF number 3

D74. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ______ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

Projection Theorem



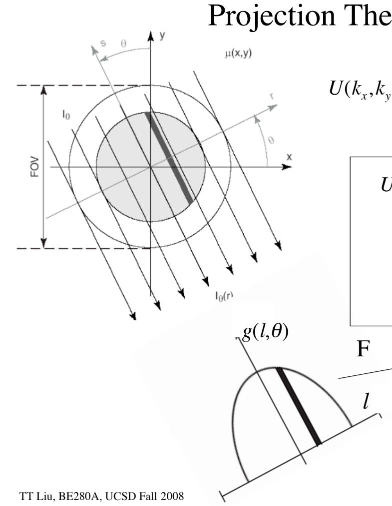
$$U(k_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dxdy$$
$$= \int_{-\infty}^{-\infty}\left[\int_{-\infty}^{-\infty} \mu(x,y) dy\right] e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{-\infty} g(x,0) e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{-\infty} g(l,0) e^{-j2\pi kl} dl$$

In-Class Example:

$$\mu(x,y) = \cos 2\pi x$$

Suetens 2002

Projection Theorem



$$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dxdy$$
$$= F_{2D}\left[\mu(x,y)\right]$$

$$U(k_x,k_y) = G(k,\theta)$$
$$k_x = k\cos\theta$$
$$k_y = k\sin\theta$$
$$k = \sqrt{k_x^2 + k_y^2}$$

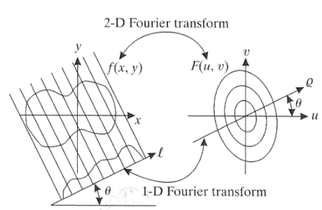
F

$$G(k,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi kl} dl$$

Suetens 2002

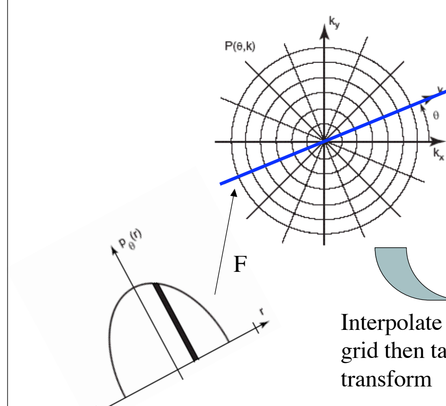
Projection Slice Theorem

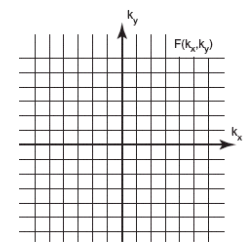
$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$



Prince&Links 2006

Fourier Reconstruction

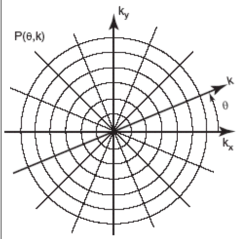




Interpolate onto Cartesian grid then take inverse transform

Suetens 2002

Polar Version of Inverse FT



$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}kdkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta + y\sin\theta)}|k|dkd\theta$$

Note :

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

Suetens 2002

Filtered Backprojection

$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}|k|dkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dkd\theta$$
$$= \int_0^{\pi} g^*(l,\theta)d\theta$$

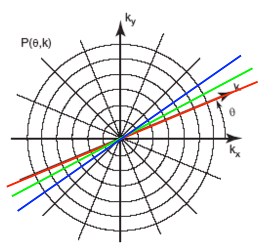
← Backproject a filtered projection

$where\ l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dk$$
$$= g(l,\theta) * F^{-1}\big[|k|\big]$$
$$= g(l,\theta) * q(l)$$

Suetens 2002

Fourier Interpretation

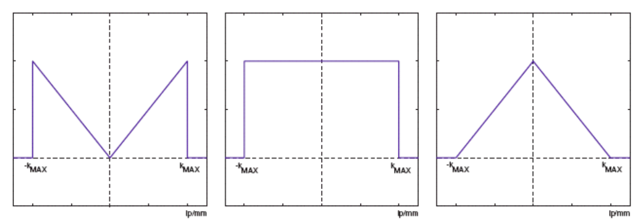


$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

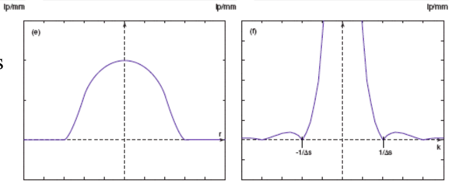
Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

Ram-Lak Filter



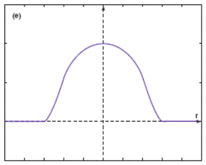
$k_{max}=1/\Delta s$



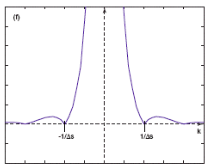
Suetens 2002

Reconstruction Path

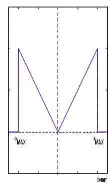
Projection



F



x

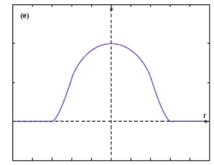


F^{-1} → Filtered Projection → Back-Project

Suetens 2002

Reconstruction Path

Projection



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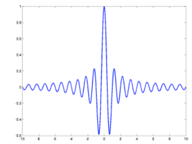


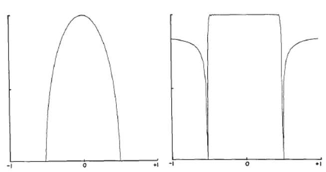
→ Filtered Projection

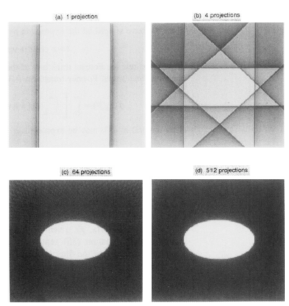
↓

Back-Project

Suetens 2002

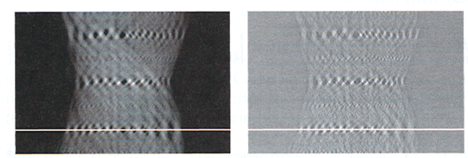
Example





Kak and Slaney

Example



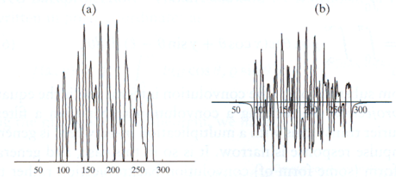


Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Prince and Links 2005

Example

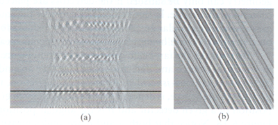


Figure 6.16
Backprojection step.

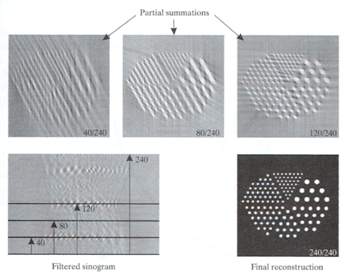


Figure 6.17
Summation step.

Prince and Links 2005