Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2008
MRI Lecture 2

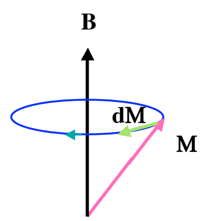
Bloch Equation

$$\frac{d\mathbf{M}}{dt} = \underbrace{\mathbf{M}\times\gamma\mathbf{B}}_{\text{Precession}} - \underbrace{\frac{M_x\mathbf{i} + M_y\mathbf{j}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0)\mathbf{k}}{T_1}}_{\text{Longitudinal Relaxation}}$$

i, j, k are unit vectors in the x,y,z directions.

Free precession about static field

$$\frac{d\mathbf{M}}{dt} = \mathbf{M}\times\gamma\mathbf{B} = \gamma\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ M_x & M_y & M_z\\ B_x & B_y & B_z\end{vmatrix} = \gamma\begin{pmatrix}\hat{i}(B_zM_y - B_yM_z)\\ -\hat{j}(B_zM_x - B_xM_z)\\ \hat{k}(B_yM_x - B_xM_y)\end{pmatrix}$$



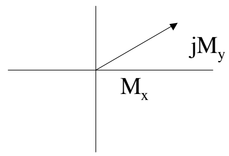
Free precession about static field

$$\begin{bmatrix}dM_x/dt\\ dM_y/dt\\ dM_z/dt\end{bmatrix} = \gamma\begin{bmatrix}B_zM_y - B_yM_z\\ B_xM_z - B_zM_x\\ B_yM_x - B_xM_y\end{bmatrix} = \gamma\begin{bmatrix}0 & B_z & -B_y\\ -B_z & 0 & B_x\\ B_y & -B_x & 0\end{bmatrix}\begin{bmatrix}M_x\\ M_y\\ M_z\end{bmatrix}$$

Precession

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_0 & 0 \\ -B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Useful to define $M \equiv M_x + jM_y$



$$dM/dt = d/dt\left(M_x + iM_y\right) = -j\gamma B_0 M$$

Solution is a time-varying phasor

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

Question: which way does this rotate with time?

Matrix Form with B=B$_0$

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \begin{bmatrix} -1/T_2 & \gamma B_0 & 0 \\ -\gamma B_0 & 1/T_2 & 0 \\ 0 & 0 & -1/T_1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0/T_1 \end{bmatrix}$$

Z-component solution

$$M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1}$$

Saturation Recovery

$$\text{If } M_z(0) = 0 \text{ then } M_z(t) = M_0(1 - e^{-t/T_1})$$

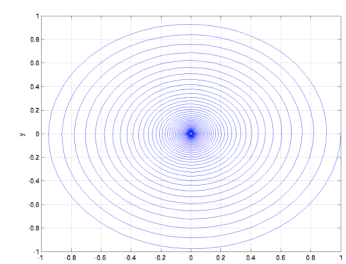
Inversion Recovery

$$\text{If } M_z(0) = -M_0 \text{ then } M_z(t) = M_0(1 - 2e^{-t/T_1})$$

Transverse Component

$M \equiv M_x + jM_y$

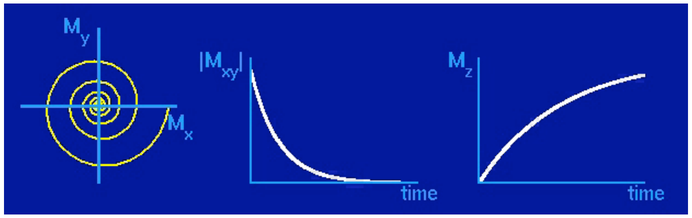
$$dM/dt = d/dt\left(M_x + iM_y\right) = -j\left(\omega_0 + 1/T_2\right)M$$



$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

Summary

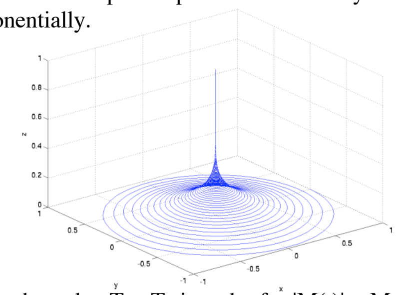
1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.



Source: http://mrsrl.stanford.edu/~brian/mri-movies/

Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.



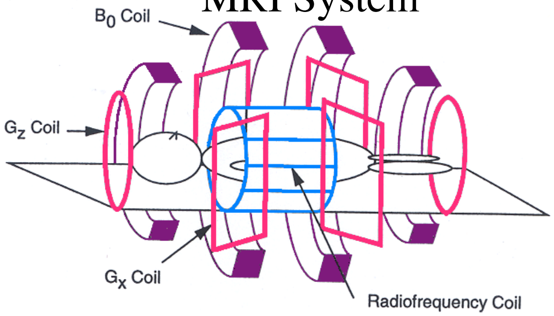
Fact: Can show that $T_2 < T_1$ in order for $|M(t)| \leq M_0$
Physically, the mechanisms that give rise to T_1 relaxation also contribute to transverse T_2 relaxation.

Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z=B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to B_z such that $B_z(x,y,z) = B_0+\Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.

MRI System



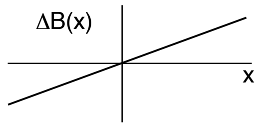
Simplified Drawing of Basic Instrumentation.
Body lies on table encompassed by
coils for static field B_o,
gradient fields (two of three shown),
and radiofrequency field B_1.

Image, caption: copyright Nishimura, Fig. 3.15

Imaging: localizing the NMR signal

RF and Gradient Coils

The local precession frequency can be changed in a position-dependent way by applying linear field gradients



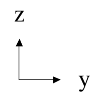
Resonant Frequency:

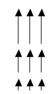
$\nu(x) = \gamma B_0 + \gamma \Delta B(x)$

Credit: R. Buxton

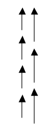
Gradient Fields

$$B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial x}x + \frac{\partial B_z}{\partial y}y + \frac{\partial B_z}{\partial z}z$$
$$= B_0 + G_x x + G_y y + G_z z$$



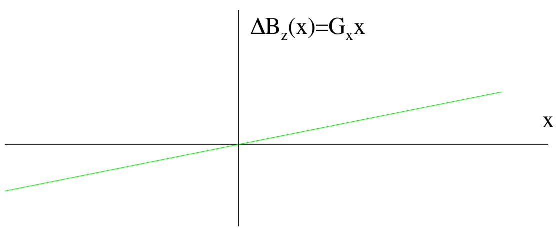


$$G_z = \frac{\partial B_z}{\partial z} > 0$$



$$G_y = \frac{\partial B_z}{\partial y} > 0$$

Interpretation



Spins Precess at $\gamma B_0 - \gamma G_x x$ (slower)

Spins Precess at γB_0

Spins Precess at at $\gamma B_0 + \gamma G_x x$ (faster)

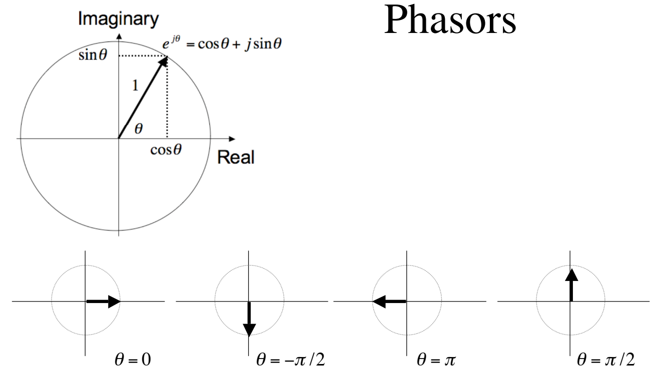
Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

Spins

There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.
Erwin Hahn

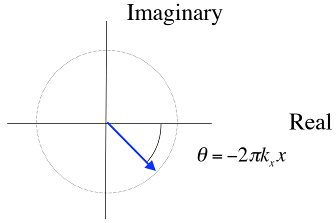
Phasors



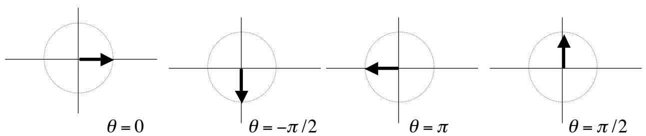
Phasor Diagram

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

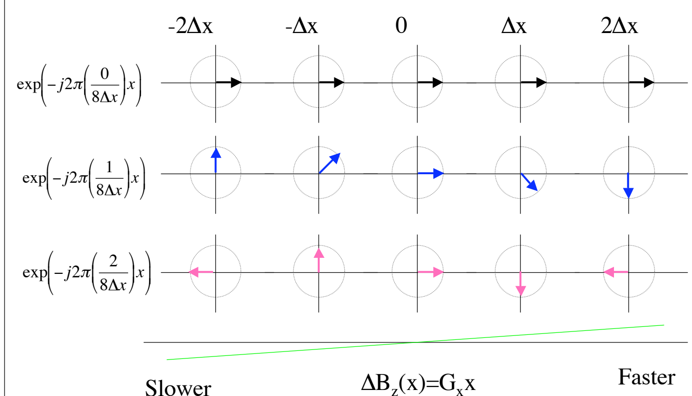
$\theta = -2\pi k_x x$

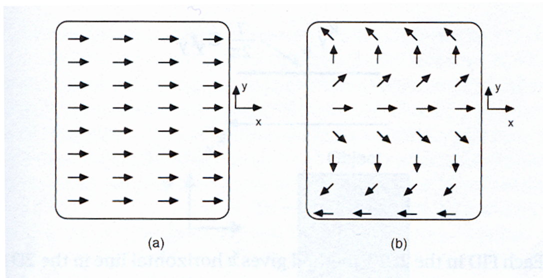


$k_x = 1;\ x = 0$	$x = 1/4$	$x = 1/2$	$x = 3/2$
$2\pi k_x x = 0$	$2\pi k_x x = \pi/2$	$2\pi k_x x = \pi$	$2\pi k_x x = 3\pi/4$



Interpretation

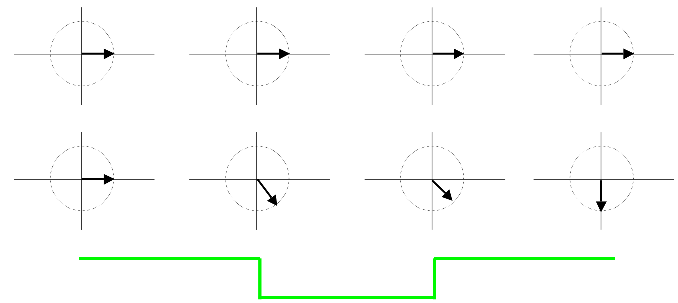




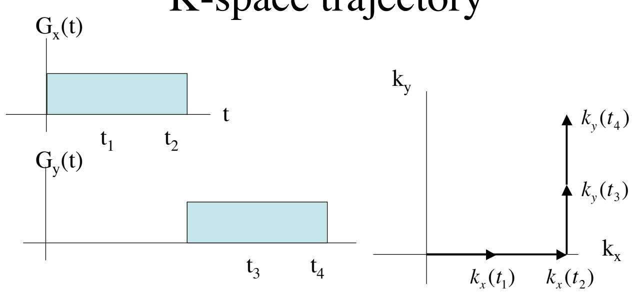
k_x=0; k_y=0 k_x=0; $k_y \neq$0

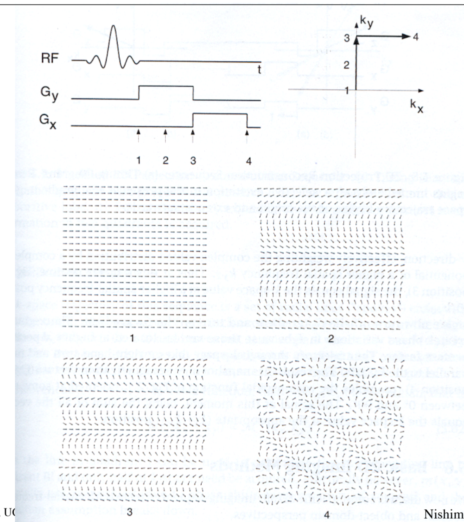
Fig 3.12 from Nishimura

Phase with time-varying gradient



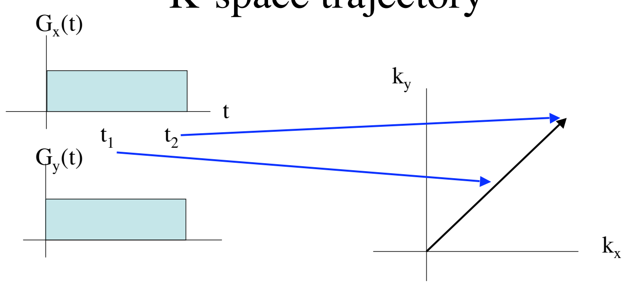
K-space trajectory





Nishimura 1996

K-space trajectory



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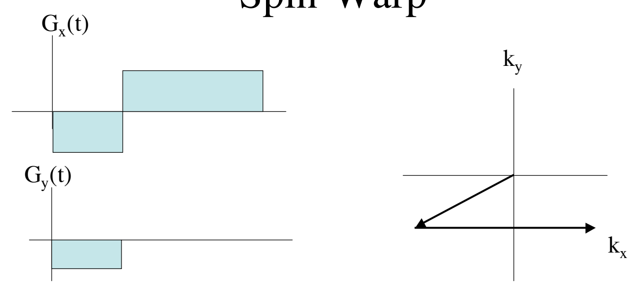
Spin-Warp



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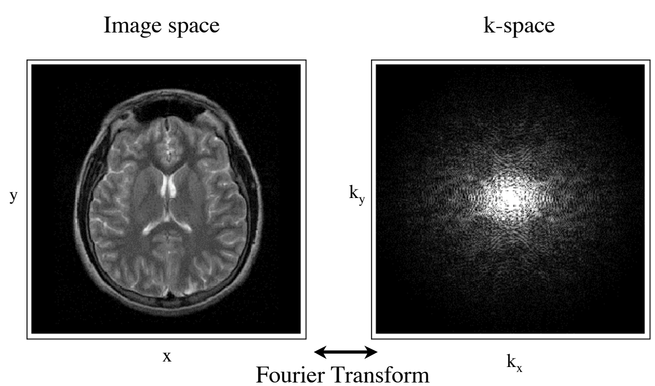
k-space



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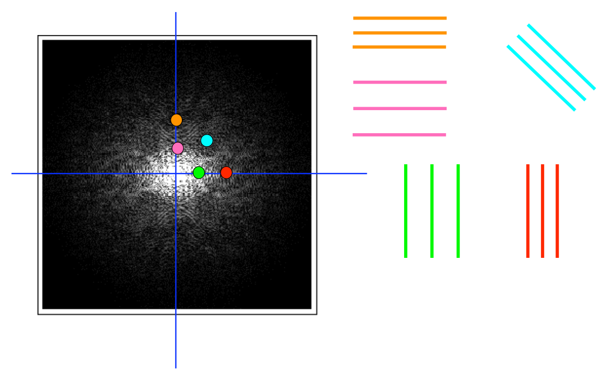

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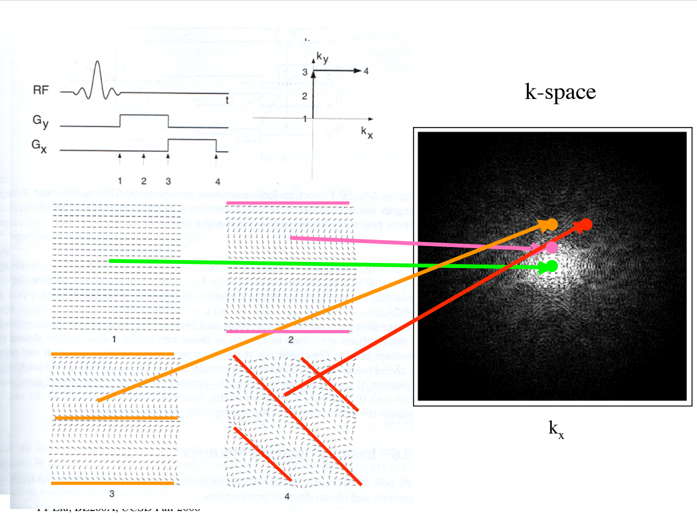
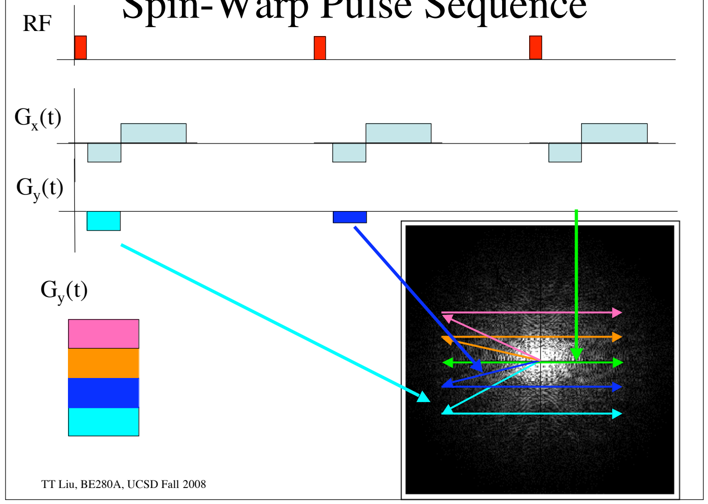
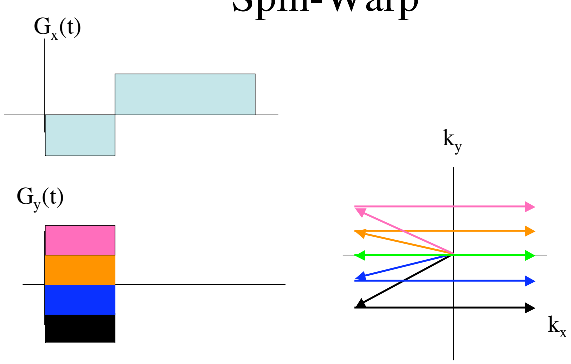


Spin-Warp Pulse Sequence



Spin-Warp



Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t) \cdot \vec{r}$$

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

$$\begin{aligned} M(\vec{r}) &= M(\vec{r},0)e^{-j\gamma B_z(\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}t}e^{-t/T_2(\vec{r})} \end{aligned}$$

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned} \omega(\vec{r},t) &= \gamma B_z(\vec{r},t) \\ &= \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r},t) \end{aligned}$$

Phase

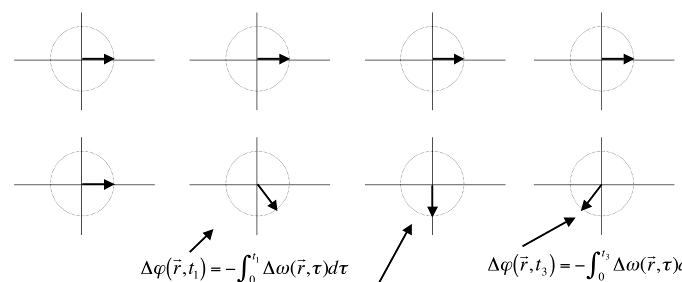
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\begin{aligned} \varphi(\vec{r},t) &= -\int_0^t \omega(\vec{r},\tau)d\tau \\ &= -\omega_0 t + \Delta\varphi(\vec{r},t) \end{aligned}$$

Where the incremental phase due to the gradients is

$$\begin{aligned} \Delta\varphi(\vec{r},t) &= -\int_0^t \Delta\omega(\vec{r},\tau)d\tau \\ &= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau \end{aligned}$$

Phase with constant gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau$$

$$\begin{aligned} \Delta\varphi(\vec{r},t_2) &= -\int_0^{t_2}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\Delta\omega(\vec{r})t_2 \end{aligned}$$

if $\Delta\omega$ is non - time varying.

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3}\Delta\omega(\vec{r},\tau)d\tau$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$M(\vec{r},t) = M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)}$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right)$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)$$

Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r},t)dV$$
$$= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$s(t) = e^{j\omega_0 t}s_r(t)$$
$$= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$
$$= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \left[G_x(\tau)x + G_y(\tau)y\right]d\tau\right)dxdy$$
$$= \int_x\int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$s(t) = \int_x\int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$
$$= M\left(k_x(t),k_y(t)\right)$$
$$= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

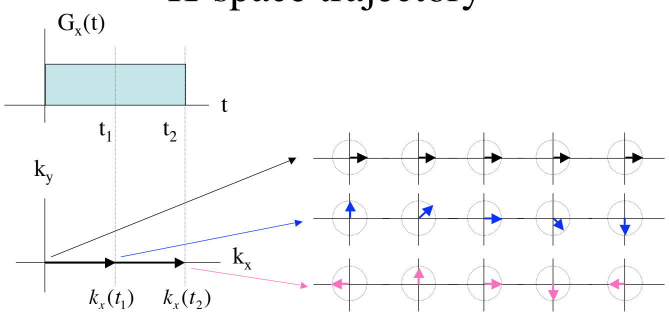
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

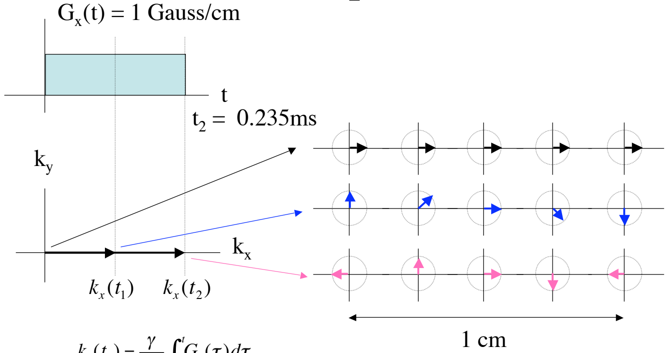
Gradient strengths have units of (magnetic field)/distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

Example



$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= 4257 Hz/G \cdot 1 G/cm \cdot 0.235 \times 10^{-3} s$$
$$= 1\ cm^{-1}$$