Basic System

Echo occurs at $t=2z/c$ where $c$ is approximately 1500 m/s or 1.5 mm/$\mu$s

Macovski 1983
Transducer

Prince and Links 2006

A-Mode (Amplitude)

Seutens 2002

M-Mode (Motion)

Seutens 2002

B-Mode (Brightness)

Seutens 2002
**B-Mode (Brightness)**

Credit: Mayo Clinic

**B-Mode**

Credit: Mayo Clinic

**CW Doppler Imaging**
PW Doppler Imaging

Color Doppler Imaging

Acoustic Waves

Speed of Sound

\[ c = \sqrt{\frac{1}{\kappa \rho}} \quad [\text{m s}^{-1}] \]

\( \kappa = \text{compressibility} \quad [\text{m}^2 \text{ kg}^{-1}] = [1/\text{Pascal}] \)

\( \rho = \text{density} \quad [\text{kg m}^{-3}] \)

<table>
<thead>
<tr>
<th>Material</th>
<th>Density</th>
<th>Speed m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1.2</td>
<td>330</td>
</tr>
<tr>
<td>Water</td>
<td>1000</td>
<td>1480</td>
</tr>
<tr>
<td>Bone</td>
<td>1380-1810</td>
<td>4080</td>
</tr>
<tr>
<td>Fat</td>
<td>920</td>
<td>1450</td>
</tr>
<tr>
<td>Liver</td>
<td>1060</td>
<td>1570</td>
</tr>
</tbody>
</table>
Acoustic Wave Equation

$$\nabla^2 p = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$

Solutions are of the form

$$p(x,t) = A_1 f_1(x - ct) + A_2 f_2(x + ct)$$

Plane Waves

$$p(z,t) = \cos(k(z - ct))$$

$$k = \text{wavenumber} = \frac{2\pi}{\lambda} = 2\pi f$$

$$\lambda = \text{wavelength} = \frac{c}{f}$$

$$f = \text{frequency} \ [\text{cycles/sec}]$$

$$T = \text{period} = \frac{1}{f}$$

Spherical Waves

Outward wave

$$p(r,t) = \frac{1}{r} \phi(t - r/c)$$

Inward wave

$$p(r,t) = \frac{1}{r} \phi(t + r/c)$$

Outward wave

$$p(r,t) = \frac{1}{r} \exp(j2\pi f (t - r/c))$$

Note: The phase depends on both space and time. At a given time, wavefront occurs at $$r = ct$$. At a given location, wavefront arrives at $$t = r/c$$.

Impedance

$$Z = \frac{P}{v} = \frac{\rho c}{\kappa}$$

$$\rho$$: density $\text{kg/m}^3$

- Brain: 1541 m/s
- Liver: 1549 m/s
- Skull bone: 4080 m/s
- Water: 1480 m/s

Note: particle velocity and speed of sound are not the same!
### Impedance

\[ Z = \rho c = \sqrt{\frac{\rho}{\kappa}} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>Density</th>
<th>Speed m/s</th>
<th>( Z ) (kg/m(^2)/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1.2</td>
<td>330</td>
<td>0.0004</td>
</tr>
<tr>
<td>Water</td>
<td>1000</td>
<td>1480</td>
<td>1.5</td>
</tr>
<tr>
<td>Bone</td>
<td>1380-1810</td>
<td>4080</td>
<td>3.75-7.38</td>
</tr>
<tr>
<td>Fat</td>
<td>920</td>
<td>1450</td>
<td>1.35</td>
</tr>
<tr>
<td>Liver</td>
<td>1060</td>
<td>1570</td>
<td>1.64-1.68</td>
</tr>
</tbody>
</table>

### Acoustic Intensity

\[ I = pv = \frac{p^2}{Z} \]

Also called acoustic energy flux. Analogous to electric power.

### Echos

- **Specular Reflection**

\[
\begin{align*}
Z_1 - P_i v_i &= Z_2 - P_r v_r \\
P_i Z_1 &= P_r Z_2 \\
P_i + P_r &= P_t \\
\rho_i Z_1 - \rho_r Z_2 = \Delta Z \\
\frac{P_i}{P_t} &= \frac{Z_2 - Z_1}{Z_2 + Z_1}
\end{align*}
\]

### Specular Reflection

<table>
<thead>
<tr>
<th>Material</th>
<th>Reflectivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain-skull</td>
<td>0.66</td>
</tr>
<tr>
<td>Fat-muscle</td>
<td>0.10</td>
</tr>
<tr>
<td>Muscle-blood</td>
<td>0.03</td>
</tr>
<tr>
<td>Soft-tissue-air</td>
<td>9999</td>
</tr>
</tbody>
</table>

\( v_i = v_r = v_t \) (velocity boundary condition)

\( P_i Z_1 = P_r Z_2 \) (pressure boundary condition)
Reflection and Refraction

Snell’s Law
\[ \frac{\sin \theta_i}{c_1} = \frac{\sin \theta_r}{c_2} = \frac{\sin \theta_t}{c_2} \]

Pressure Reflectivity
\[ R = \frac{p_r}{p_i} = \frac{Z_2 \cos \theta_r - Z_1 \cos \theta_i}{Z_2 \cos \theta_r + Z_1 \cos \theta_i} \]

Pressure Transmittivity
\[ T = \frac{p_t}{p_i} = \frac{2Z_2 \cos \theta_t}{Z_2 \cos \theta_r + Z_1 \cos \theta_i} \]

Example: Fat/liver interface at normal incidence
\[ Z_{\text{fat}} = 1.35 \times 10^{-6} \text{ kg m}^{-2} \text{ s}^{-1} \]
\[ Z_{\text{liver}} = 1.66 \times 10^{-6} \text{ kg m}^{-2} \text{ s}^{-1} \]
\[ R_I = \left( \frac{Z_{\text{liver}} - Z_{\text{fat}}}{Z_{\text{liver}} + Z_{\text{fat}}} \right)^2 = 0.103 \]
Scattering

Point scatterers retransmit the incident wave equally in all directions (e.g., isotropic scattering).

Attenuation


\[ p(z,t) = A_z f(t - c/z) = A_0 \exp(-\mu_a z) f(t - c/z) \]

Amplitude attenuation factor

\[ \mu_a = -\frac{\ln A_z}{z} : \text{units = nepers/cm} \]

\[ \alpha = -20 \frac{\ln A_z}{z} A_0 = 20 \mu_a \log_{10}(e) = 8.7 \mu_a : \text{dB/cm} \]

Attenuation coefficient

Example

Example: Fat at 5 MHz

Attenuation coefficient = 5 MHz \times 0.63 \text{ dB/cm/MHz} = 3.15 \text{ dB/cm}

After 4 cm, attenuation = 4 \times 3.15 = 12.6 \text{ dB}

Relative amplitude is \(10^{(3.15/20)} = 0.2344\)

Recall dB = 20\log_{10}(A_z/A_0)

<table>
<thead>
<tr>
<th>Material</th>
<th>(\alpha_0) [dB/cm/MHz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>fat</td>
<td>0.63</td>
</tr>
<tr>
<td>liver</td>
<td>0.94</td>
</tr>
<tr>
<td>Cardiac muscle</td>
<td>1.8</td>
</tr>
<tr>
<td>bone</td>
<td>20.0</td>
</tr>
</tbody>
</table>
Received signal

Attenuation Correction

\[ e(t) = K \int \int \int \frac{e^{-2\alpha}}{z} R(x,y,z) s(x,y) p(t-2z/c) \, dx \, dy \, dz \]

\[ \approx K \int \int \int \alpha c \tau e^{\frac{ct}{2}} R(x,y,z) s(x,y) p(t-\frac{2\tau}{c}) \, dx \, dy \, d\tau \]

Attenuation Correction and Signal Equation
Depth Response

Depth response
\[
p(t-2z_0/c) = p(2z/c - 2z_0/c) = p\left(\frac{2(z-z_0)}{c}\right)
\]

Therefore impulse response is simply
\[
p(t) \quad \text{in the time domain or}
\]
\[
p(2z/c) \quad \text{in the spatial domain}
\]

Depth Resolution

Depth resolution
\[
\Delta z = \frac{cT}{2} \approx 1.5 \frac{c}{f_0} = 1.5\lambda.
\]

Example:
For \(f_0 = 5 \text{ MHz}\), \(\lambda = c/f_0 = (1500 \text{ m/s})/(5 \times 10^6 \text{ Hz}) = 0.3 \text{ mm}\)
\(\Delta z = 1.5\lambda = 0.45 \text{ mm}\)

Trade-off
Higher \(f_0\) ⇒ Smaller \(\Delta z\) ⇒ but more attenuation

Example:
Assume 1 dB/cm/MHz
For 10 cm depth, 20 cm roundtrip path length.
At 1 MHz 20 dB of attenuation ⇒ Attenuation = 0.1
At 10 MHz 200 dB of attenuation ⇒ Attenuation = 1x10^{-10}
**Depth of Penetration**

Assume system can handle $L$ dB of loss, then

$$L = 20 \log \left( \frac{A_z}{A_0} \right)$$

We also have the definition

$$\alpha = \frac{1}{L} \cdot 20 \log \left( \frac{A_z}{A_0} \right)$$

and the approximation

$$\alpha = \alpha_0 f$$

Total range a wave can travel before attenuation $L$ is

$$z = \frac{L}{\alpha_0 f}$$

Depth of penetration is

$$d_p = \frac{L}{2 \alpha_0 f}$$

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**Pulse Repetition and Frame Rate**

Need to wait for echoes to return before transmitting new pulse

Pulse repetition interval is

$$T_R \geq \frac{2 d_p}{c}$$

Pulse repetition rate is

$$f_R = \frac{1}{T_R}$$

If $N$ pulses are required to form an image, then the frame rate is

$$F = \frac{1}{N T_R}$$

---

**Example**

$N = 256, L = 80$ dB, $c = 1540 \text{ m/s}$, $\alpha_0 = 1 \text{ dB/cm/MHz}$

What frequency should be used to achieve a frame rate of 15 frame/sec?

$$T_R = \frac{1}{F} = 0.26 \text{ ms}$$

$$T_R \geq \frac{2 d_p}{c} = \frac{L}{\alpha_0 f c}$$

$$f \geq \frac{L}{\alpha_0 c T_R} = 1.99 \text{ MHz}$$
Single-slit Diffraction

Source: wikipedia

Square Aperture Diffraction

Source: wikipedia

Huygen’s Principle

Source: wikipedia

Huygen’s Principle

\[ k = \frac{2\pi}{\lambda} \]

Aperture function
\[ U(x_1, y_1) \]

Field amplitude
\[ U(x_0, y_0) \]

Wavenumber
\[ k = \frac{2\pi}{\lambda} \]

Obliquity Factor
\[ \theta_{01} \]

where
\[ h(P_0, P_1) = \frac{\exp(jk\rho_{01})}{\rho_{01}} \cos(\theta^*, \theta_{01}) \]

Anderson and Trahey 2000
Two-Slit Interference

Plane Wave (Fraunhofer) Approximation

$\sin \theta = \frac{x_0}{r_0} = \frac{x_0}{z}$

$\sin \theta = \frac{x_0}{r_0} = \frac{x_0}{z}$

$\Delta z \approx \frac{x_0}{z}$

Plane Wave Approximation

In general

$U(x_0, y_0) = \frac{1}{z} \exp \left( j \frac{2\pi}{\lambda} \left( z - \frac{x_0 x_1}{z} \right) \right)$

Example

$s(x, y) = \text{rect}(x/D) \text{rect}(y/D)$

$U(x_0, y_0) = \frac{1}{z} \exp \left( j \frac{2\pi}{\lambda} \left( \frac{Dx}{\lambda} \sin \left( \frac{Dk_x}{\lambda} \right) \sin \left( \frac{Dk_y}{\lambda} \right) \right) \right)$

Zeros occur at $x_0 = \frac{n\lambda}{D}$ and $y_0 = \frac{n\lambda}{D}$

Beamwidth of the sinc function is $\frac{\lambda}{D}$
Example

$$\text{rect}\left(\frac{x}{D}\right) \ast \text{comb}\left(\frac{x}{d}\right) \Leftrightarrow \text{Sidelobes}$$

Question: What should we do to reduce the sidelobes?