Simplified Drawing of Basic Instrumentation.
Body lies on table encompassed by coils for static field $B_0$, gradient fields (two of three shown), and radiofrequency field $B_1$. Image, caption: copyright Nishimura, Fig. 3.15

RF Excitation

At equilibrium, net magnetization is parallel to the main magnetic field. How do we tip the magnetization away from equilibrium?

$B_1$ radiofrequency field tuned to Larmor frequency and applied in transverse (xy) plane induces nutation (at Larmor frequency) of magnetization vector as it tips away from the z-axis. Lab frame of reference.
RF Excitation

\[ B_1(t) = 2B_1(t)\cos(\omega t)i \]

\[ = B_1(t)(\cos(\omega t)i - \sin(\omega t)j) + B_1(t)(\cos(\omega t)i + \sin(\omega t)j) \]

Nishimura 1996.
Rotating Frame Bloch Equation

\[ \frac{d M_{\text{rot}}}{dt} = M_{\text{rot}} \times \gamma B_{\text{eff}} \]

\[ B_{\text{eff}} = B_{\text{rot}} + \omega_{\text{rot}} \gamma = B_1(t)\hat{i} + B_0\hat{k} \]

Note: we use the RF frequency to define the rotating frame. If this RF frequency is on-resonance, then the main B0 field doesn’t cause any precession in the rotating frame. However, if the RF frequency is off-resonance, then there will be a net precession in the rotating frame that is give by the difference between the RF frequency and the local Larmor frequency.

Let \( B_{\text{rot}} = B_1(t)\hat{i} + B_0\hat{k} \)

\[ B_{\text{eff}} = B_{\text{rot}} + \frac{\omega_{\text{rot}}}{\gamma} \]

\[ = B_1(t)\hat{i} + \left( B_0 - \frac{\omega}{\gamma} \right)\hat{k} \]

If \( \omega = \omega_0 \)

\[ = \gamma B_0 \]

Then \( B_{\text{eff}} = B_1(t)\hat{i} \)
Flip angle
\[ \theta = \int_0^r \omega_1(s) ds \]
\[ \omega_1(t) = \gamma B_1(t) \]
Let $B_{rot} = B_1(t)i + (B_0 + \gamma G_z)k$

$$B_{eff} = B_{rot} + \frac{\omega_{rot}}{\gamma}$$

$$= B_1(t)i + (B_0 + \gamma G_z - \frac{\omega}{\gamma})k$$

If $\omega = \omega_0$

$$B_{eff} = B_1(t)i + (\gamma G_z)k$$

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**Small Tip Angle Approximation**

![Diagram](image)

*For small $\theta$*

$$M_z = M_0 \cos \theta \approx M_0$$

$$M_{xy} = M_0 \sin \theta \approx M_0 \theta$$

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**Excitation k-space**

At each time increment of width $\Delta \tau$, the excitation $B_1(\tau)$ produces an increment in magnetization of the form $\Delta M_{xy} = JM_{xy}(\tau)\Delta \tau$

(small tip angle approximation)

In the presence of a gradient, this will accumulate phase of the form $\phi = -\gamma \int G_z(s)ds$, such that the incremental magnetization at time $t$ is

$$\Delta M_{xy}(t; z; \tau) = JM_{xy}(\tau)\exp\left(-j\frac{\gamma}{\omega} \int G_z(s)ds\right)\Delta \tau$$

Integrating over all time increments, we obtain

$$M_{xy}(t; z) = JM_{xy} \int G_z(s)ds \exp\left(-j\frac{\gamma}{\omega} \int G_z(s)ds\right)\Delta \tau$$

$$= JM_{xy} \int G_z(s)ds \exp\left(j2\pi k(t; \tau)z\right)\Delta \tau$$

where $k(t; \tau) = -\frac{\gamma}{2\pi} \int G_z(s)ds$

Pauly et al 1989
Excitation k-space

\[ M_\omega(t,z) = jM_0 \int_{-\tau}^{\tau} \gamma B_i(\tau) \exp(j2\pi k(\tau, t)z) d\tau \]

This has the form of an inverse Fourier transform, where we are integrating the contributions of the field \( B_i(\tau) \) at the k-space point \( k(\tau, t) \).

Small Tip Angle Example

\[ B_i(t) = B_{rect} \left( \frac{\tau}{T} \right) \]

\[ M_i(\tau/2, z) = jM_0 \exp(-j\omega_z \tau/2) F^{-1} \left[ jM_0 \exp(-j\omega_z \tau/2) F \left[ \frac{\gamma G}{2\pi} \right] \right] \]

For a constant gradient:

\[ k_i(t, r) = \frac{1}{2\pi} G_z (r - t) \]

\[ d\tau = \frac{2\pi}{\gamma G_z} dk_z \]

\[ M_\omega(t, z) = jM_0 \int_{-\tau}^{\tau} \gamma B_i(\tau) \exp(j2\pi k(\tau, t)z) d\tau \]

\[ = jM_0 \int_{-\tau}^{\tau} \gamma B_i(\tau) \exp(j2\pi k_0 z) \frac{2\pi}{\gamma G_z} dk_z \]

\[ = jM_0 \exp(-j\omega_z t) F^{-1} \left[ j\gamma B_i(k_z) \right] \]

Refocusing

\[ M_r(\tau, z) = \exp(j\omega_z t/2) M_i(\tau, z) \]

\[ = jM_0 \exp(j\omega_z t/2) \exp(-j\omega_z t/2) F^{-1} \left[ j\gamma B_i(k_z) \right] \]

For a constant gradient:

\[ k_r(t, r) = \frac{1}{2\pi} G_z (r - t) \]

\[ d\tau = \frac{2\pi}{\gamma G_z} dk_z \]

\[ M_r(t, z) = jM_0 \int_{-\tau}^{\tau} \gamma B_i(\tau) \exp(j2\pi k(\tau, t)z) d\tau \]

\[ = jM_0 \int_{-\tau}^{\tau} \gamma B_i(\tau) \exp(j2\pi k_0 z) \frac{2\pi}{\gamma G_z} dk_z \]

\[ = jM_0 \exp(-j\omega_z t) F^{-1} \left[ j\gamma B_i(k_z) \right] \]

Nishimura 1996
Refocusing

\[ M_\text{o}(t,z) = jM_0 \int_{-\infty}^{\infty} B_1(\tau) \exp(j2\pi k(\tau,t)z) d\tau \]

This has the form of an inverse Fourier transform, where we are integrating the contributions of the field \( B_1(\tau) \) at the \( k \)-space point \( k(\tau,t) \).

Slice Selection

\[ k(\tau,t) = \frac{\gamma G_z z}{2\pi} \]

Small Tip Angle Example

\[ B_1(t) = \text{Asinc}(t/\tau) \left( 0.5 + 0.46\cos\left(\frac{2\pi}{\tau}\right) \right) \]

\[ F^{-1}(B_1(k_z)) = \text{rect} \left( \frac{2\gamma G_z z}{\tau} \right) W \left( \frac{\gamma G_z z}{2\pi} \right) \]

First zero in \( k_z \) space is at \( \frac{\gamma G_z z}{2\pi} \)

Therefore, width of the rect function is \( \Delta z = \frac{2\pi}{\gamma G_z z} \)
Slice Selection

Example
\( \Delta z = 5 \text{ mm}; \tau = 400 \mu\text{sec}; \theta = \pi/2 \)

\[
G_z = \frac{2\pi}{\gamma \Delta z \tau} \left( \frac{1}{4257 \text{Hz}/(G)(0.5 \text{cm})(400e-6)} \right) = 1.175 \text{ G/cm}
\]

\[
\theta = \gamma \int_0^\infty B_i \sin \left( \frac{s - T/2}{\tau} \right) ds \approx \gamma B_i \cdot \text{(area of sinc)} = \gamma B_i \tau
\]

\[
B_i = \frac{\theta}{\gamma \tau} \left( \frac{\pi/2}{2\pi(4257 \text{Hz}/G)(400e-6)} \right) = 0.1468 \text{ G}
\]

Time-Bandwidth Product (TBW)
\[
\text{sinc}(t/\tau) \ast \text{rect}(fT) \approx \text{rect}(fT) \ast 2N \gamma \text{sinc}(2Nf)
\]

Duration = \( 2N\tau \)

Bandwidth = \( \frac{1}{\tau} \) ⇒ \( \Delta f = \frac{2\pi}{\gamma \Delta z \tau} \)

Transition Width = \( \frac{1}{2N\tau} \) ⇒ \( \Delta \tau' = \frac{2\pi}{\gamma \Delta z \tau} \)

Time – Bandwidth Product (TBW) = \( 2N\tau \frac{1}{\tau} = 2N \)

also, \( \text{TBW} = \frac{\text{Bandwidth}}{\text{Transition Width}} \)

For a fixed duration pulse, we can increase TBW by increasing the Bandwidth.
(Note: this will also lead to an increase in N). This will require a higher B1 amplitude and a higher gradient to keep the slice width constant. Note that with higher TBW the physical transition width then decreases.

(Note 1996)

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Cardiac Tagging

Excitation k-space

Multi-dimensional Excitation k-space

\[ M_x(t, \mathbf{r}) = jM_0 \int_{-\infty}^{t} \omega_0(\tau) \exp \left( -j\int_{\tau}^{t} G(s) \cdot \mathbf{r} ds \right) d\tau \]

\[ = jM_0 \int_{-\infty}^{t} \omega_0(\tau) \exp \left( j2\pi \mathbf{k}(\tau) \cdot \mathbf{r} \right) d\tau \]

where \( \mathbf{k}(\tau) = -\frac{\gamma}{2\pi} \int_{\tau}^{t} G(\tau') dt' \)

Pauly et al 1989

Excitation k-space

Panich MRM 1999