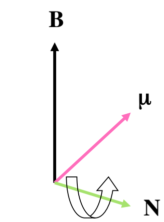
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2012
MRI Lecture 2

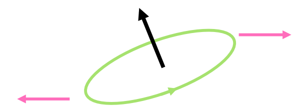
Torque



$\mathbf{N} = \mu \times \mathbf{B}$

Torque

For a non-spinning magnetic moment, the torque will try to align the moment with magnetic field (e.g. compass needle)



Precession

Torque

$\mathbf{N} = \mu \times \mathbf{B}$

$\frac{d\mathbf{S}}{dt} = \mathbf{N}$

Change in Angular momentum

$\frac{d\mathbf{S}}{dt} = \mu \times \mathbf{B}$

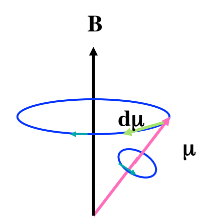
$\mu = \gamma \mathbf{S}$

Relation between magnetic moment and angular momentum

$\frac{d\mu}{dt} = \mu \times \gamma\mathbf{B}$

Precession

$\frac{d\mu}{dt} = \mu \times \gamma\mathbf{B}$

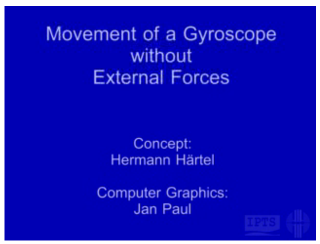


Analogous to motion of a gyroscope

Precesses at an angular frequency of

$\omega = \gamma B$

This is known as the **Larmor** frequency.



http://www.astrophysik.uni-kiel.de/~hhaertelmpg_e/gyros_free.mpg

Magnetization Vector

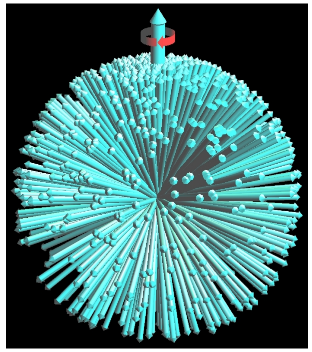
Vector sum of the magnetic moments over a volume.

For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

Equation of motion is the same form as for individual moments.

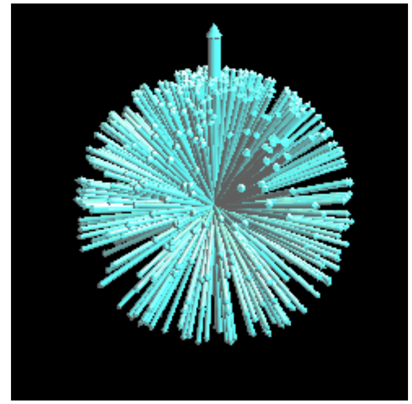
$$\mathbf{M} = \frac{1}{V} \sum_{\substack{\text{protons} \\ \text{in V}}} \mu_i$$

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}$$



Hansen 2009

RF Excitation



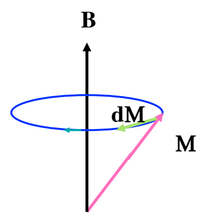
http://www.drcmr.dk/main/content/view/213/74/

Free precession about static field

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}$$

$$= \gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \gamma \begin{pmatrix} \hat{i}\left(B_z M_y - B_y M_z\right) \\ -\hat{j}\left(B_z M_x - B_x M_z\right) \\ \hat{k}\left(B_y M_x - B_x M_y\right) \end{pmatrix}$$



Free precession about static field

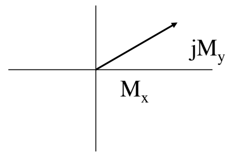
$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} B_z M_y - B_y M_z \\ B_x M_z - B_z M_x \\ B_y M_x - B_x M_y \end{bmatrix}$$

$$= \gamma \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Precession

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_0 & 0 \\ -B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Useful to define $M \equiv M_x + jM_y$



$$dM/dt = d/dt\left(M_x + iM_y\right)$$
$$= -j\gamma B_0 M$$

Solution is a time-varying phasor

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

Question: which way does this rotate with time?

Gyromagnetic Ratios

Nucleus	Spin	Magnetic Moment	$\gamma/(2\pi)$ (MHz/Tesla)	Abundance
^{1}H	1/2	2.793	42.58	88 M
^{23}Na	3/2	2.216	11.27	80 mM
^{31}P	1/2	1.131	17.25	75 mM

Source: Haacke et al., p. 27

Larmor Frequency

$\omega = \gamma B$ Angular frequency in rad/sec

$f = \gamma B / (2\pi)$ Frequency in cycles/sec or Hertz, Abbreviated Hz

For a 1.5 T system, the Larmor frequency is 63.86 MHz which is 63.86 million cycles per second. For comparison, KPBS-FM transmits at 89.5 MHz.

Note that the earth's magnetic field is about 50 μT, so that a 1.5T system is about 30,000 times stronger.

Notation and Units

1 Tesla = 10,000 Gauss

Earth's field is about 0.5 Gauss

0.5 Gauss = 0.5×10^{-4} T = 50 μT

$\gamma = 26752$ radians/second/Gauss

$\gamma\!\!\!^{-} = \gamma / 2\pi = 4258$ Hz/Gauss

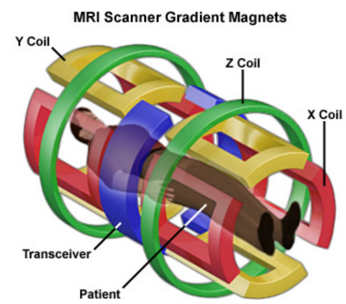
$= 42.58$ MHz/Tesla

Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z=B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to B_z such that $B_z(x,y,z) = B_0+\Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.

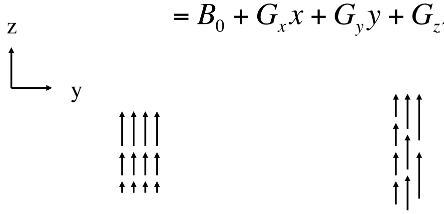
MRI Gradients



http://www.magnet.fsu.edu/education/tutorials/magnetacademy/mri/fullarticle.html

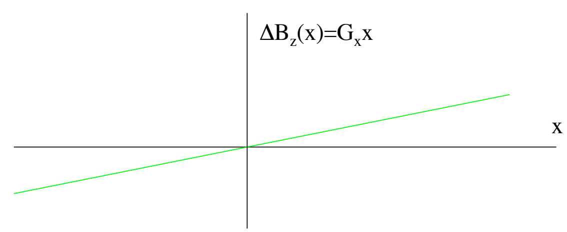
Gradient Fields

$$B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial x}x + \frac{\partial B_z}{\partial y}y + \frac{\partial B_z}{\partial z}z$$
$$= B_0 + G_x x + G_y y + G_z z$$



$$G_z = \frac{\partial B_z}{\partial z} > 0 \qquad G_y = \frac{\partial B_z}{\partial y} > 0$$

Interpretation



Spins Precess at $\gamma B_0 - \gamma G_x x$ (slower)

$$M(t) = M(0)e^{-j\gamma(B_0 - G_x x)t} = M(0)e^{-j(\omega_0 - \Delta\omega)t}$$

Spins Precess at γB_0

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

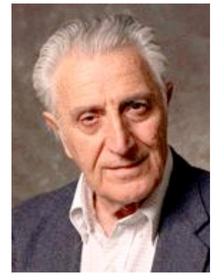
Spins Precess at at $\gamma B_0 + \gamma G_x x$ (faster)

$$M(t) = M(0)e^{-j\gamma(B_0 + G_x x)t} = M(0)e^{-j(\omega_0 + \Delta\omega)t}$$

Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

Spins



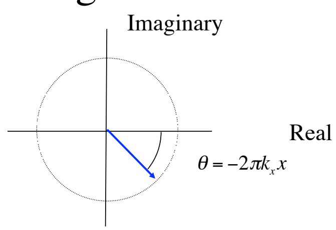
There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.
Erwin Hahn

TT. Liu, BE280A, UCSD Fall 2012

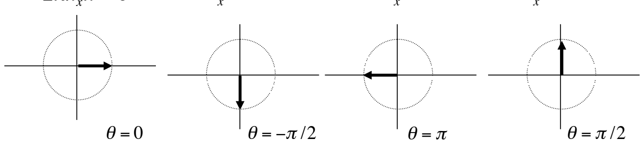
Phasor Diagram

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$\theta = -2\pi k_x x$

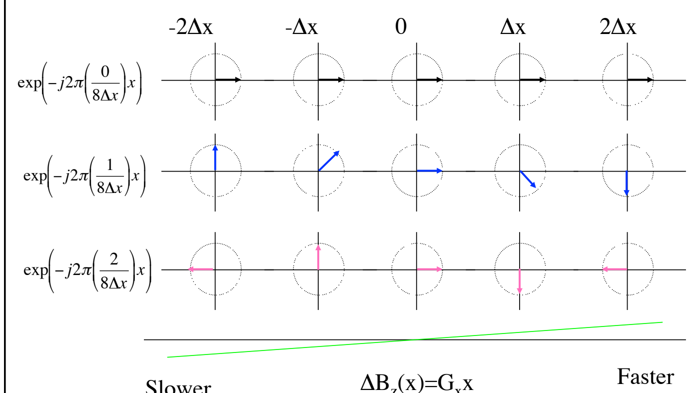


$k_x = 1;\ x = 0$	$x = 1/4$	$x = 1/2$	$x = 3/2$
$2\pi k_x x = 0$	$2\pi k_x x = \pi/2$	$2\pi k_x x = \pi$	$2\pi k_x x = 3\pi/4$

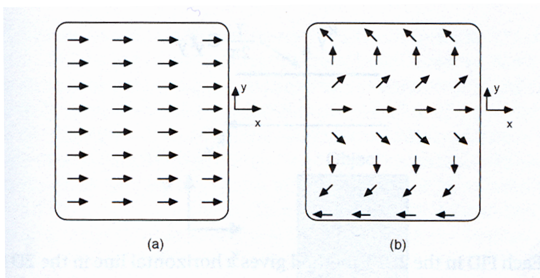


TT. Liu, BE280A, UCSD Fall 2012

Interpretation

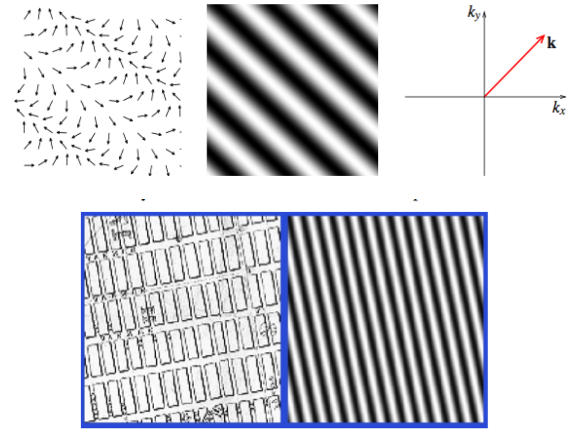


TT. Liu, BE280A, UCSD Fall 2012

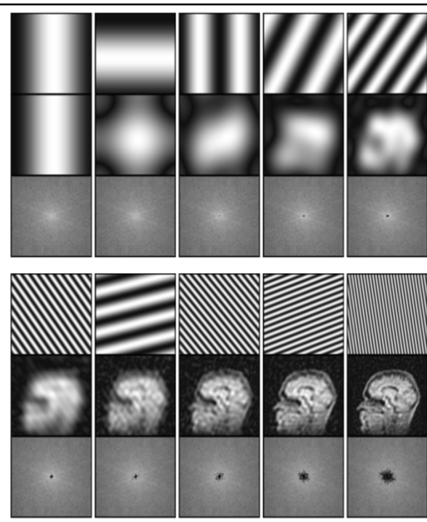


$k_x=0; k_y=0$ $k_x=0; k_y\neq 0$

Fig 3.12 from Nishimura

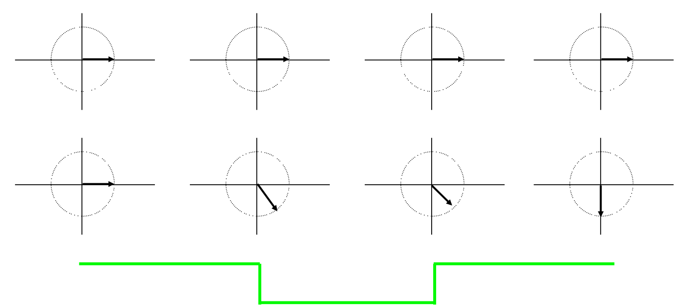


Hanson 2009

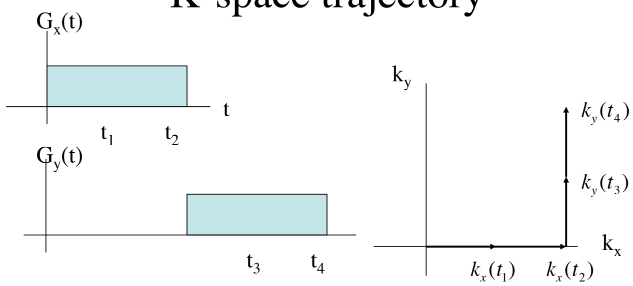


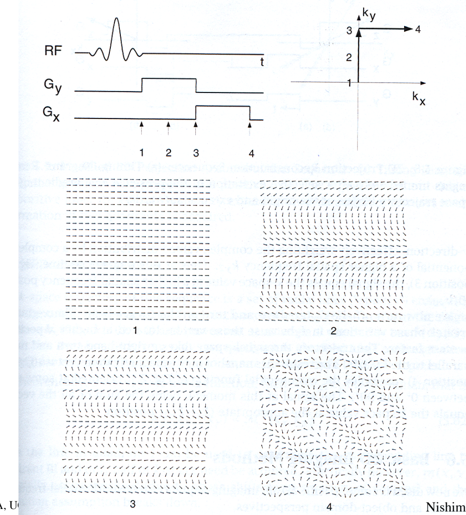
Hanson 2009

Phase with time-varying gradient

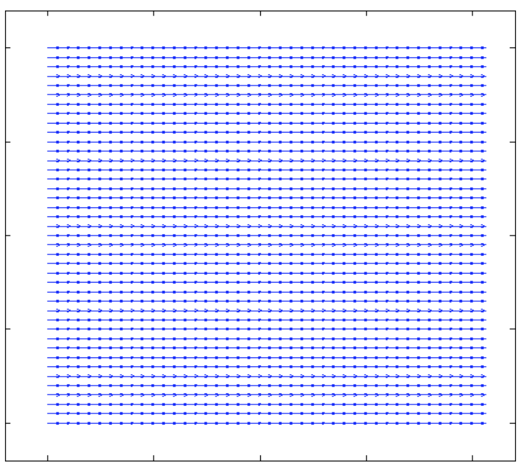


K-space trajectory

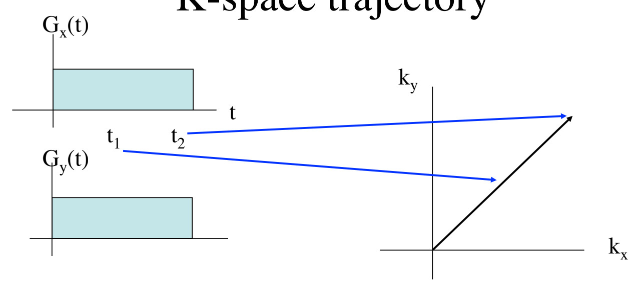




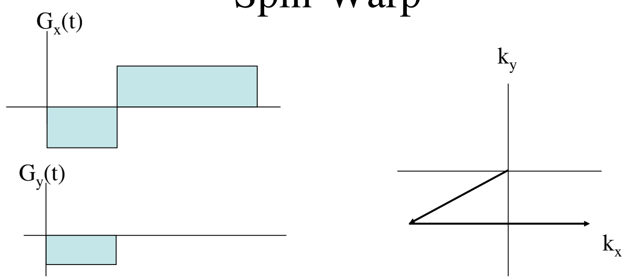
Nishimura 1996



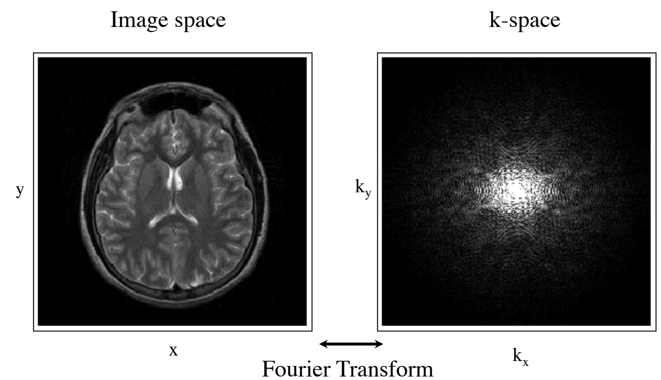
K-space trajectory

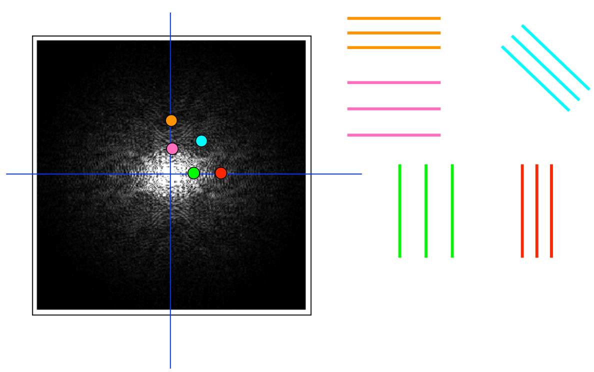


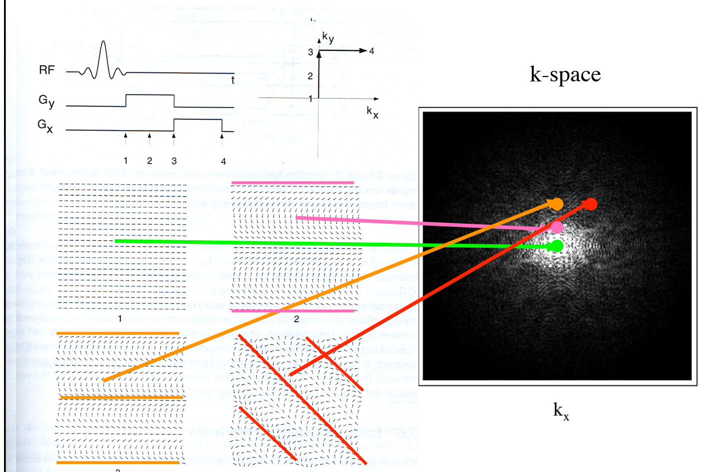
Spin-Warp



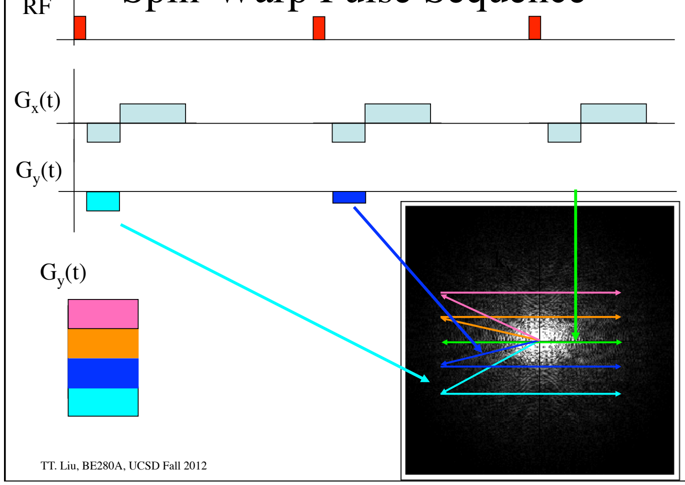
k-space





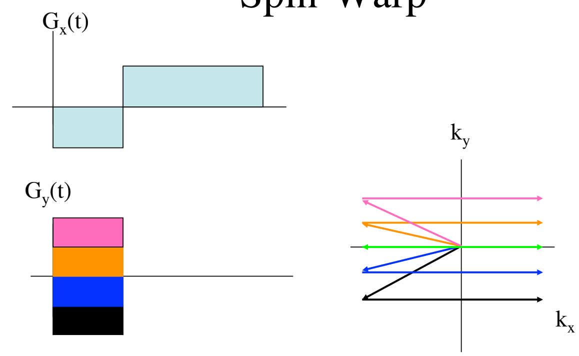


Spin-Warp Pulse Sequence

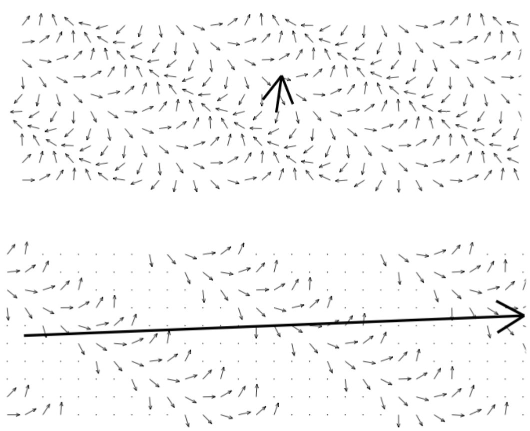


TT. Liu, BE280A, UCSD Fall 2012

Spin-Warp

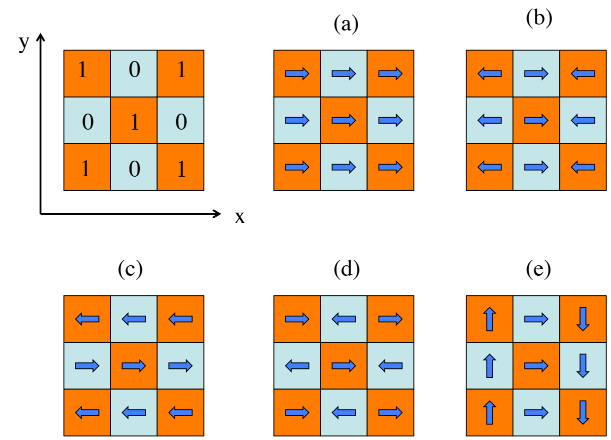


TT. Liu, BE280A, UCSD Fall 2012



Hanson 2009

TT. Liu, BE280A, UCSD Fall 2012



TT. Liu, BE280A, UCSD Fall 2012

Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t) \cdot \vec{r}$$

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

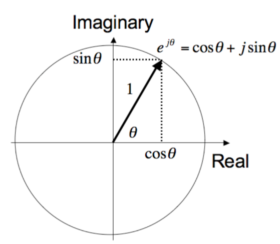
$$\begin{aligned} M(\vec{r}) &= M(\vec{r},0)e^{-j\gamma B_z(\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}t}e^{-t/T_2(\vec{r})} \end{aligned}$$

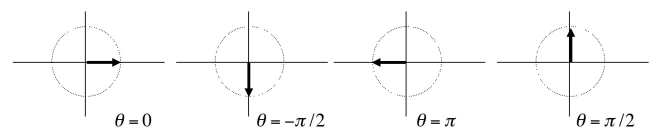
Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned} \omega(\vec{r},t) &= \gamma B_z(\vec{r},t) \\ &= \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r},t) \end{aligned}$$

Phasors





Phase

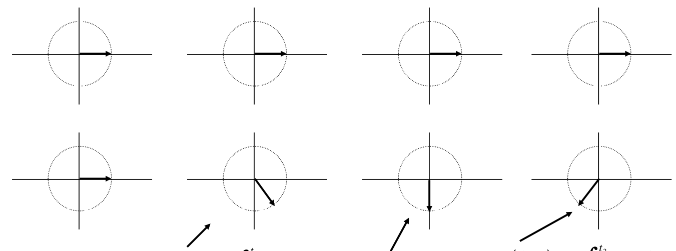
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\varphi(\vec{r},t) = -\int_0^t \omega(\vec{r},\tau)d\tau$$
$$= -\omega_0 t + \Delta\varphi(\vec{r},t)$$

Where the incremental phase due to the gradients is

$$\Delta\varphi(\vec{r},t) = -\int_0^t \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau$$

Phase with constant gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1} \Delta\omega(\vec{r},\tau)d\tau$$

$$\Delta\varphi(\vec{r},t_2) = -\int_0^{t_2} \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\Delta\omega(\vec{r})t_2$$
if $\Delta\omega$ is non - time varying.

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3} \Delta\omega(\vec{r},\tau)d\tau$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$M(\vec{r},t) = M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)}$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right)$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)$$

Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r},t)dV$$
$$= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$
\begin{aligned}
s(t) &= e^{j\omega_0 t} s_r(t) \\
&= \int_x \int_y m(x,y) \exp\left(-j\gamma \int_o^t \vec{G}(\tau) \cdot \vec{r}\, d\tau\right) dxdy \\
&= \int_x \int_y m(x,y) \exp\left(-j\gamma \int_o^t \left[G_x(\tau)x + G_y(\tau)y\right] d\tau\right) dxdy \\
&= \int_x \int_y m(x,y) \exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy
\end{aligned}
$$

Where

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$
\begin{aligned}
s(t) &= \int_x \int_y m(x,y) \exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy \\
&= M\left(k_x(t), k_y(t)\right) \\
&= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}
\end{aligned}
$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

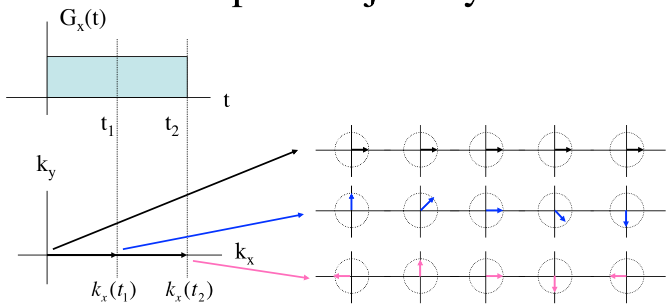
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

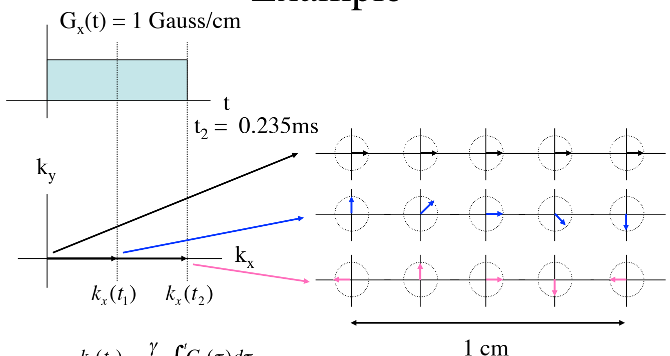
Gradient strengths have units of (magnetic field)/ distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

Example



$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= 4257 Hz/G \cdot 1 G/cm \cdot 0.235\times10^{-3} s$$
$$= 1\ cm^{-1}$$