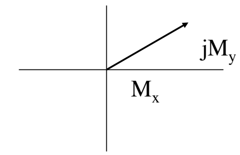
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2012
MRI Lecture 3

Precession

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_0 & 0 \\ -B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Useful to define $M \equiv M_x + jM_y$



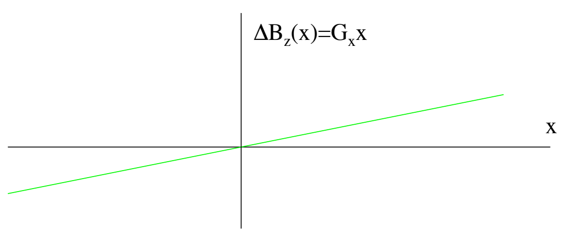
$$dM/dt = d/dt\left(M_x + iM_y\right) = -j\gamma B_0 M$$

Solution is a time-varying phasor

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

Question: which way does this rotate with time?

Interpretation



Spins Precess at γB_0- $\gamma G_x x$ (slower)

$$M(t) = M(0)e^{-j\gamma(B_0 - G_x x)t} = M(0)e^{-j(\omega_0 - \Delta\omega)t}$$

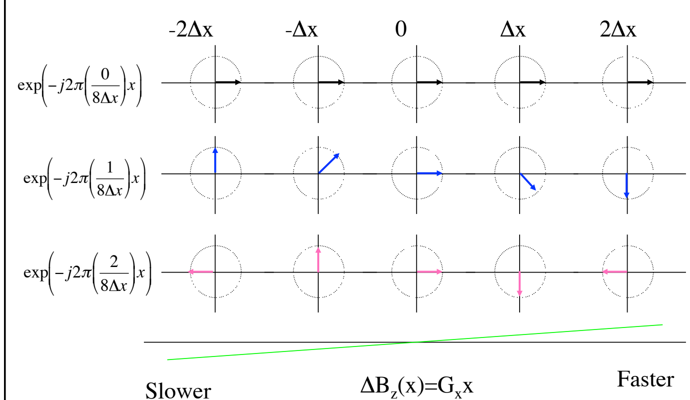
Spins Precess at γB_0

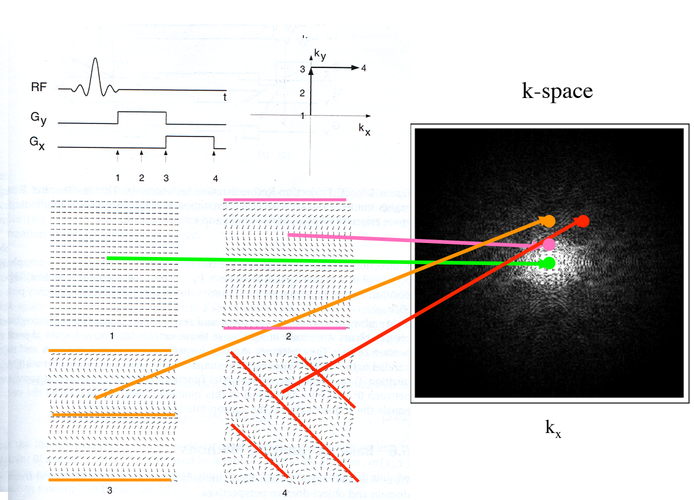
$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$

Spins Precess at at γB_0+ $\gamma G_x x$ (faster)

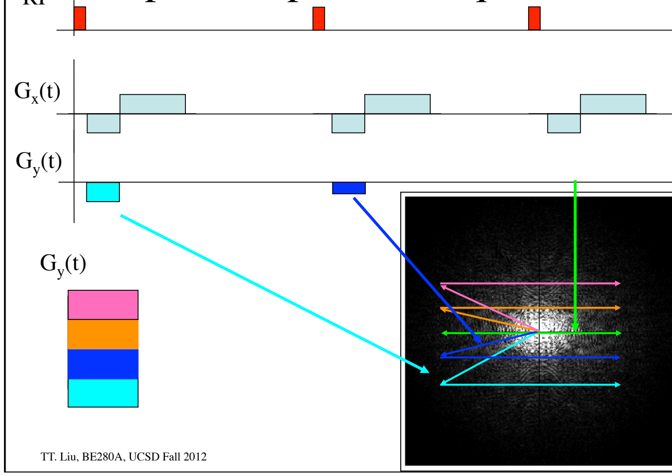
$$M(t) = M(0)e^{-j\gamma(B_0 + G_x x)t} = M(0)e^{-j(\omega_0 + \Delta\omega)t}$$

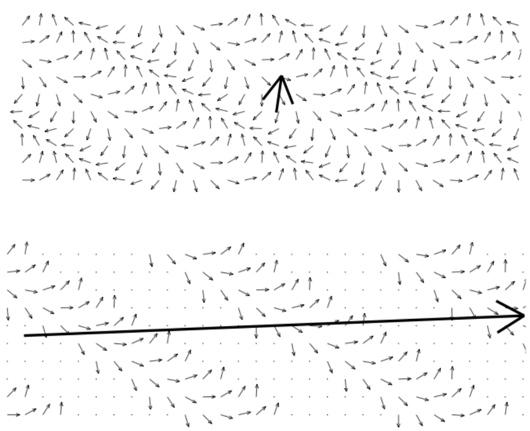
Interpretation



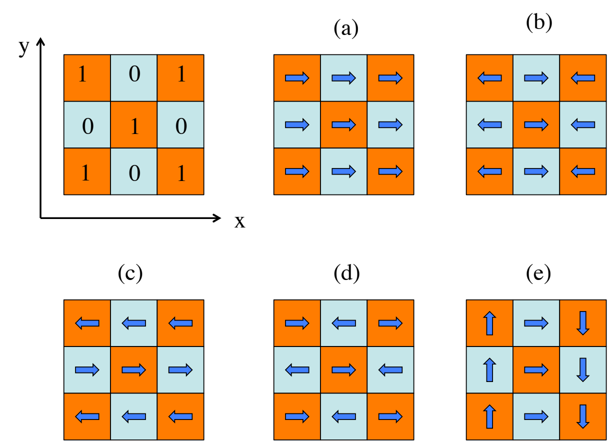


Spin-Warp Pulse Sequence





Hanson 2009



Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G}\cdot\vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t)\cdot\vec{r}$$

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

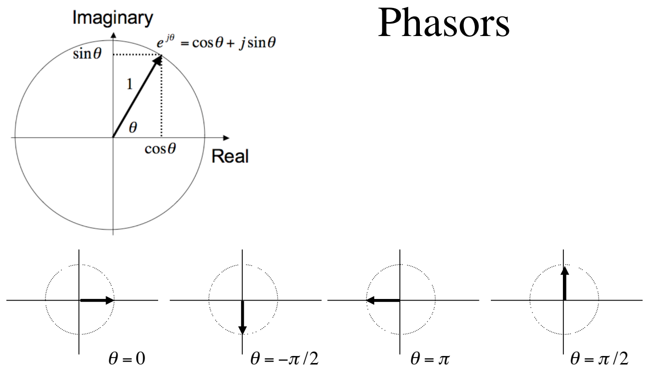
$$\begin{aligned} M(\vec{r}) &= M(\vec{r},0)e^{-j\gamma B_z(\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}t}e^{-t/T_2(\vec{r})} \end{aligned}$$

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned} \omega(\vec{r},t) &= \gamma B_z(\vec{r},t) \\ &= \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r},t) \end{aligned}$$

Phasors



Phase

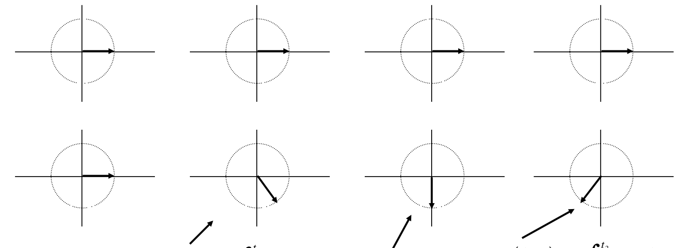
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\varphi(\vec{r},t) = -\int_0^t \omega(\vec{r},\tau)d\tau$$
$$= -\omega_0 t + \Delta\varphi(\vec{r},t)$$

Where the incremental phase due to the gradients is

$$\Delta\varphi(\vec{r},t) = -\int_0^t \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau$$

Phase with constant gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1} \Delta\omega(\vec{r},\tau)d\tau$$

$$\Delta\varphi(\vec{r},t_2) = -\int_0^{t_2} \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\Delta\omega(\vec{r})t_2$$
if $\Delta\omega$ is non - time varying.

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3} \Delta\omega(\vec{r},\tau)d\tau$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$M(\vec{r},t) = M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)}$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right)$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)$$

Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r},t)dV$$
$$= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dx\,dy\,dz$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dx\,dy$$

Signal Equation

Demodulate the signal to obtain

$$
\begin{aligned}
s(t) &= e^{j\omega_0 t} s_r(t) \\
&= \int_x \int_y m(x,y) \exp\left(-j\gamma \int_o^t \vec{G}(\tau)\cdot \vec{r}\, d\tau\right) dxdy \\
&= \int_x \int_y m(x,y) \exp\left(-j\gamma \int_o^t \left[G_x(\tau)x + G_y(\tau)y\right] d\tau\right) dxdy \\
&= \int_x \int_y m(x,y) \exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy
\end{aligned}
$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$
\begin{aligned}
s(t) &= \int_x \int_y m(x,y) \exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy \\
&= M\left(k_x(t), k_y(t)\right) \\
&= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}
\end{aligned}
$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

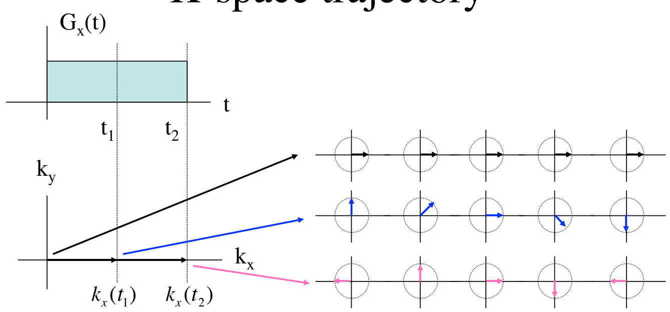
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

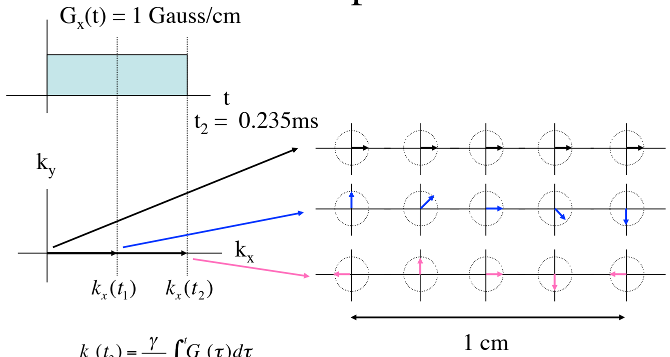
Gradient strengths have units of (magnetic field)/ distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

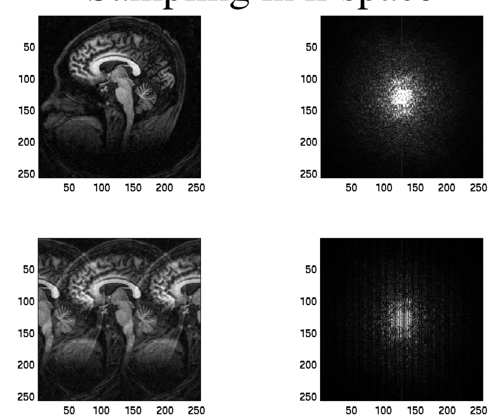
$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

Example

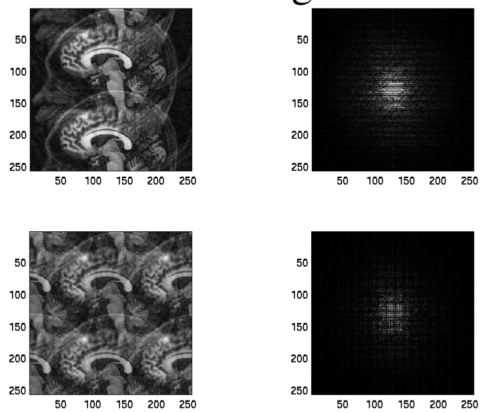


$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= 4257 Hz/G \cdot 1 G/cm \cdot 0.235\times10^{-3} s$$
$$= 1\ cm^{-1}$$

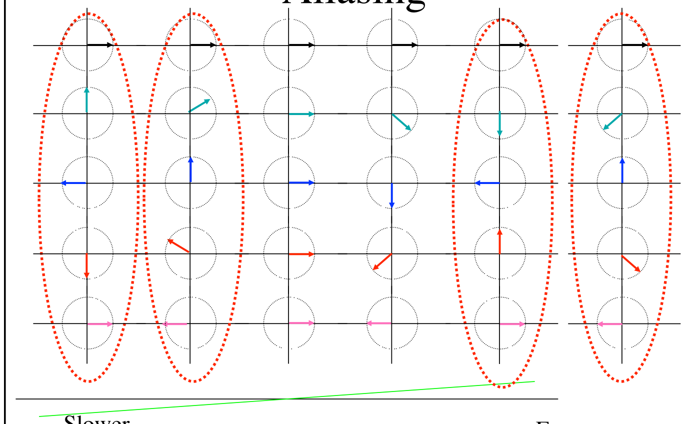
Sampling in k-space



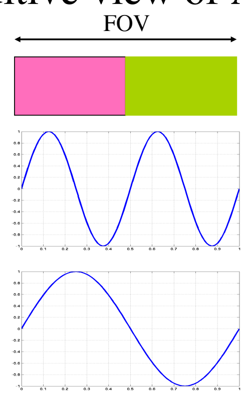
Aliasing



Aliasing



Intuitive view of Aliasing

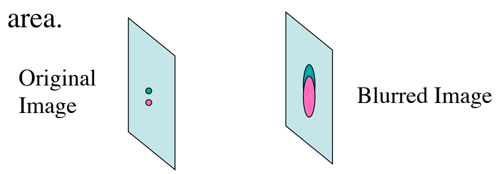


$k_x = 2/FOV$

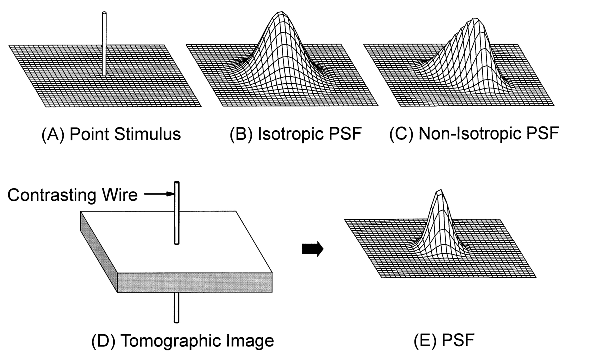
$k_x = 1/FOV$

Impulse Response

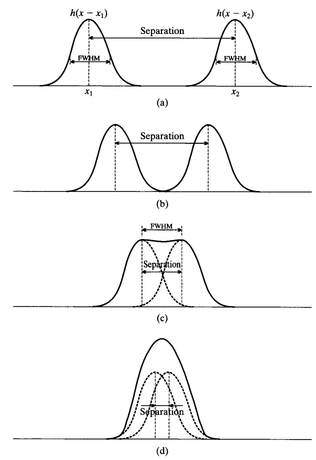
Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).



Bushberg et al 2001

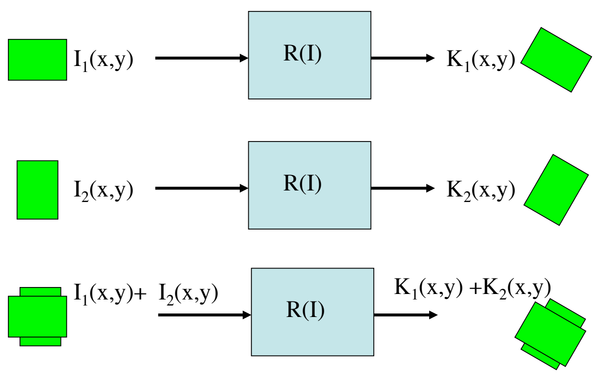


Full Width Half Maximum (FWHM) is a measure of resolution.

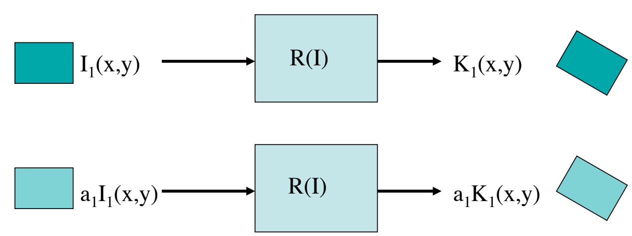
Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

Prince and Link 2005

Linearity (Addition)



Linearity (Scaling)



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

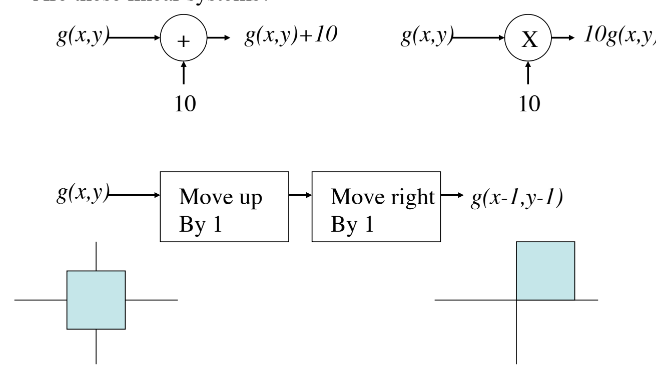
$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$

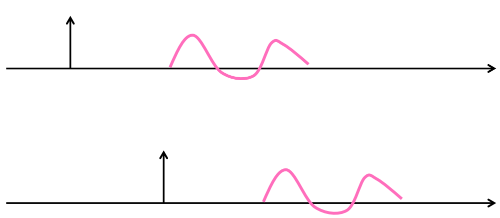
Example

Are these linear systems?



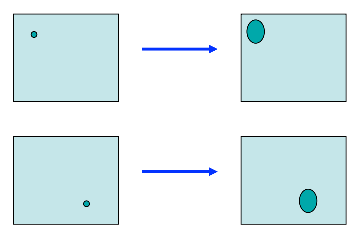
Time Invariance

If a system is time invariant, the impulse response depends only on the difference between the time of the output and the position of the impulse and is given by $h(t_2;t)=h(t_2-t)$

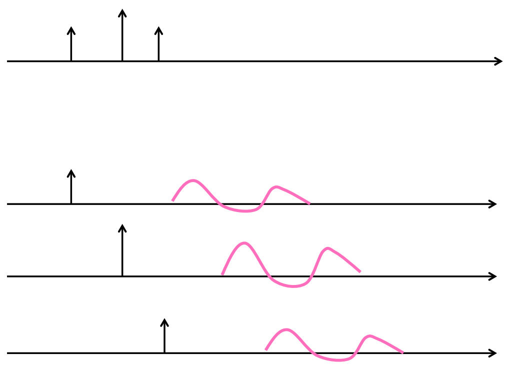


Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta)=h(x_2-\xi,y_2-\eta)$



LTI Response



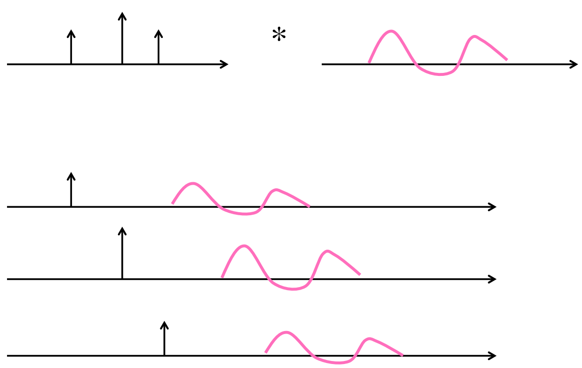
1D Convolution

$$I(x) = \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi$$
$$= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$
$$= g(x) * h(x)$$

Useful fact:

$$g(x) * \delta(x-\Delta) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi$$
$$= g(x-\Delta)$$

LTI Response



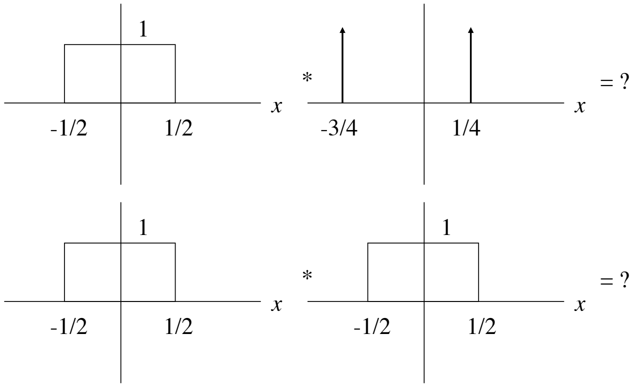
2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta$$
$$= g(x_2,y_2) ** h(x_2,y_2)$$

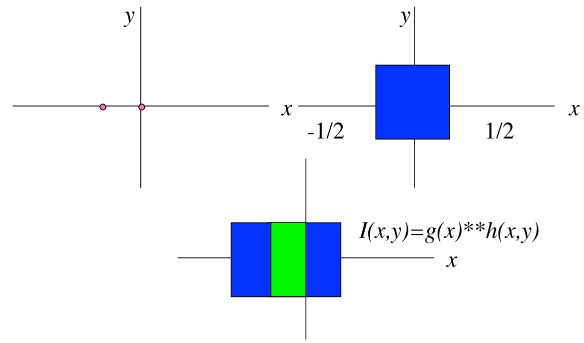
where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2)= g(x_2, y_2)*h(x_2, y_2)$.

1D Convolution Examples

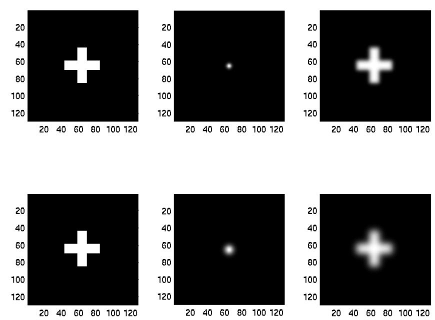


2D Convolution Example

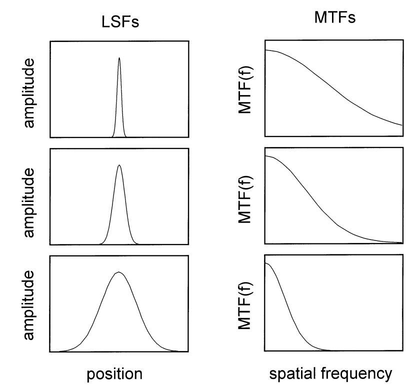
$g(x) = \delta(x+1/2,y) + \delta(x,y)$ $h(x)=rect(x,y)$



2D Convolution Example

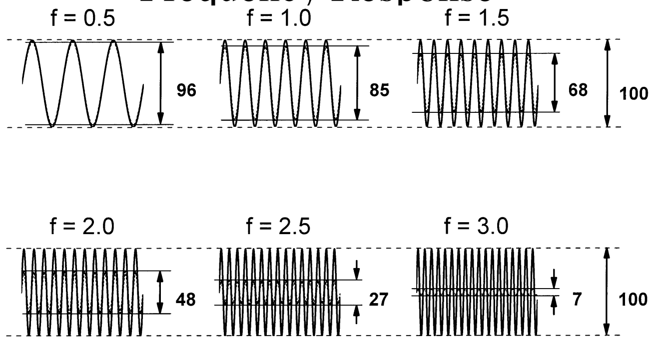


MTF = Fourier Transform of Impulse Response



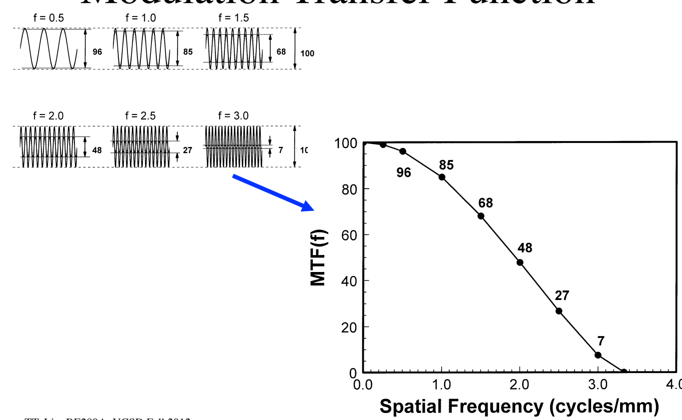
Bushberg et al 2001

Modulation Transfer Function (MTF) or Frequency Response



Bushberg et al 2001

Modulation Transfer Function



Bushberg et al 2001

Figure 1:

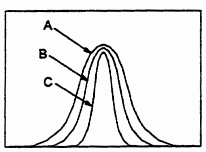
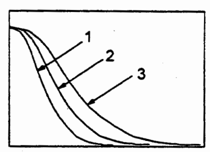


Figure 2:



8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSF), which LSF will yield the best spatial resolution?

10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

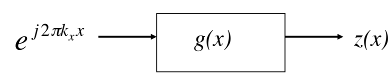
A. MTF number 1
B. MTF number 2
C. MTF number 3

D74. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ______ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

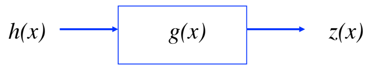


$$
\begin{aligned}
z(x) &= g(x) * e^{j2\pi k_x x} \\
&= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du \\
&= G(k_x) e^{j2\pi k_x x}
\end{aligned}
$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

Convolution/Multiplication

Now consider an arbitrary input *h(x)*.



Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u) * h(x-u) du\right] e^{-j2\pi k_x x} dx \\ &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\ &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\ &= G(k_x) H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

2D Convolution/Multiplication

Convolution

$$F[g(x,y) ** h(x,y)] = G(k_x,k_y) H(k_x,k_y)$$

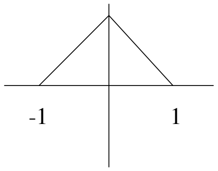
Multiplication

$$F[g(x,y) h(x,y)] = G(k_x,k_y) ** H(k_x,k_y)$$

Application of Convolution Thm.

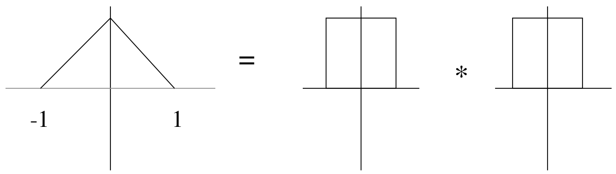
$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1} (1-|x|) e^{-j2\pi k_x x} dx = ??$$

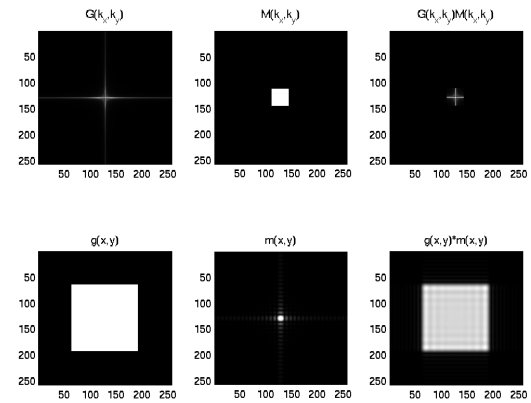


Application of Convolution Thm.

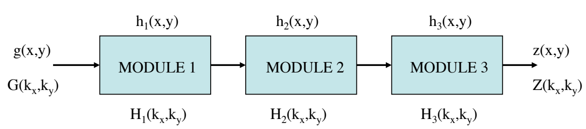
$$\Lambda(x) = \Pi(x) * \Pi(x)$$
$$F(\Lambda(x)) = \mathrm{sinc}^2(k_x)$$



Convolution Example



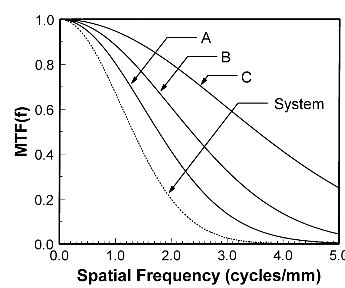
Response of an Imaging System



$z(x,y)=g(x,y)**h_1(x,y)**h_2(x,y)**h_3(x,y)$

$Z(k_x,k_y)=G(k_x,k_y)\ H_1(k_x,k_y)\ H_2(k_x,k_y)\ H_3(k_x,k_y)$

System MTF = Product of MTFs of Components



Bushberg et al 2001

Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots FWHM_N^2}$$

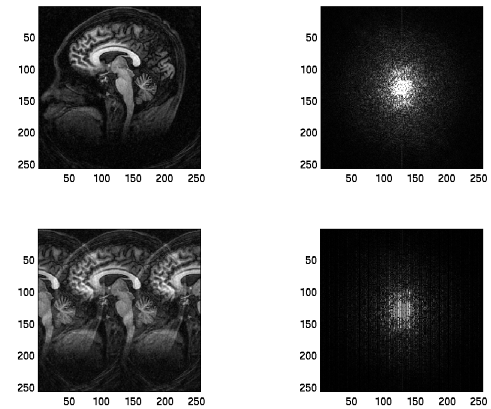
Example

$$FWHM_1 = 1mm$$

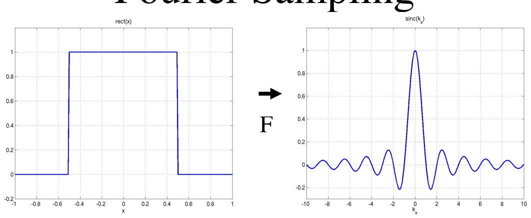
$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$

Sampling in k-space

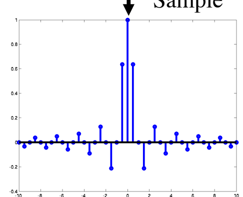


Fourier Sampling



F

Sample

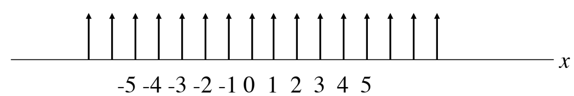


Instead of sampling the signal, we sample its Fourier Transform

??? F^{-1}

Comb Function

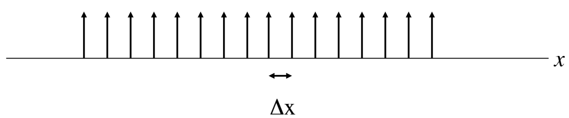
$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$
$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$
$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

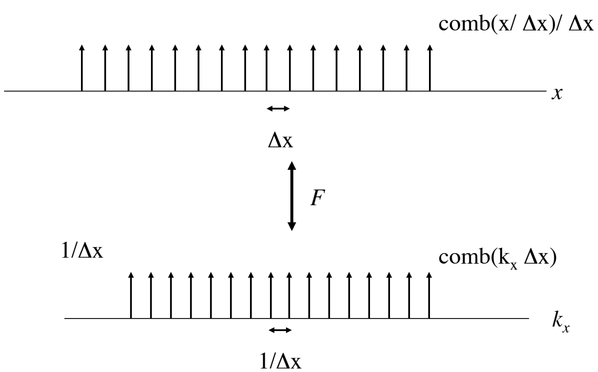


Fourier Transform of comb(x)

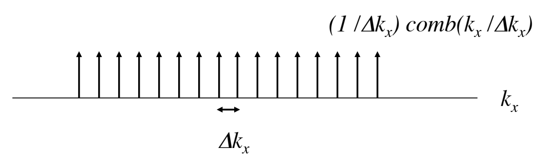
$$F\left[comb(x)\right] = comb(k_x)$$
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)\right] = \frac{1}{\Delta x} \Delta x\, comb(k_x \Delta x)$$
$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$
$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{n}{\Delta x}\right)$$

Fourier Transform of comb(x/ Δx)

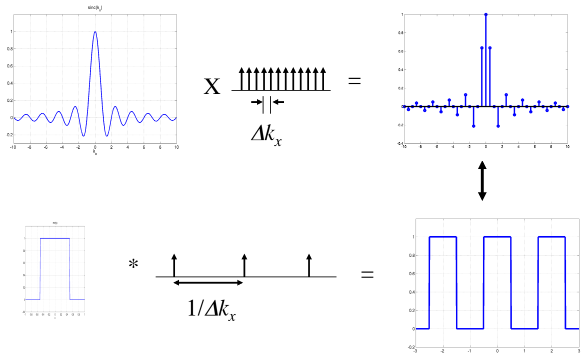


Fourier Sampling

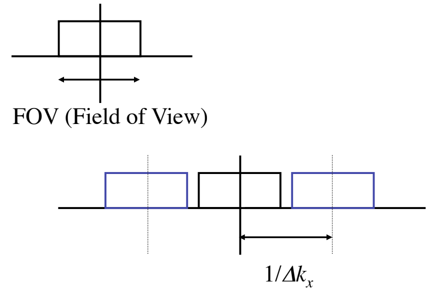


$$G_S(k_x) = G(k_x) \frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)$$
$$= G(k_x) \sum_{n=-\infty}^{\infty} \delta(k_x - n\Delta k_x)$$
$$= \sum_{n=-\infty}^{\infty} G(n\Delta k_x) \delta(k_x - n\Delta k_x)$$

Fourier Sampling -- Inverse Transform

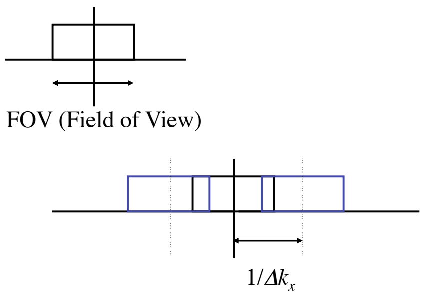


Nyquist Condition



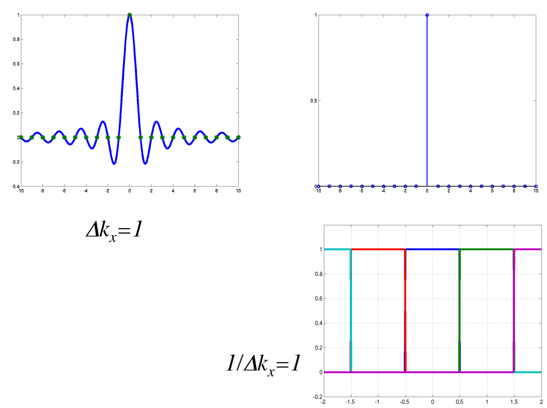
To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/\text{FOV}$

Aliasing



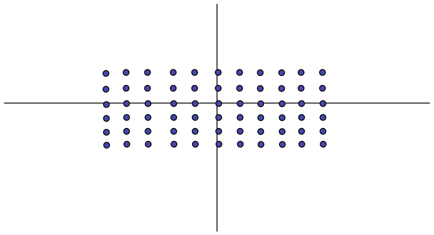
Aliasing occurs when $1/\Delta k_x <$ FOV

Aliasing Example



2D Comb Function

$$comb(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m)\delta(y-n)$$
$$= comb(x)comb(y)$$



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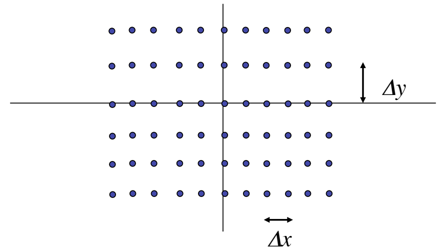
Scaled 2D Comb Function

$$comb(x/\Delta x, y/\Delta y) = comb(x/\Delta x)comb(y/\Delta y)$$
$$= \Delta x\Delta y\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x)\delta(y-n\Delta y)$$



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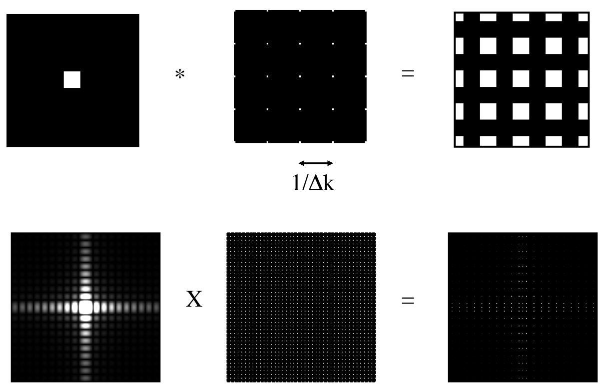

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2D k-space sampling

$$G_S(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y}comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)$$
$$= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x-m\Delta k_x,k_y-n\Delta k_y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}G(m\Delta k_x,n\Delta k_y)\delta(k_x-m\Delta k_x,k_y-n\Delta k_y)$$

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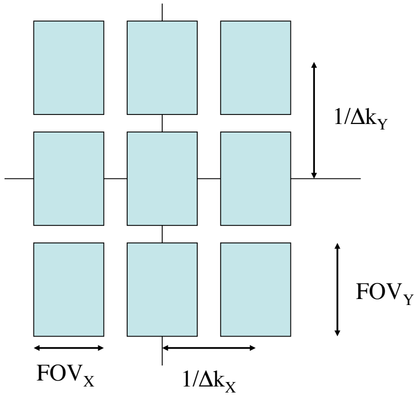
Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

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