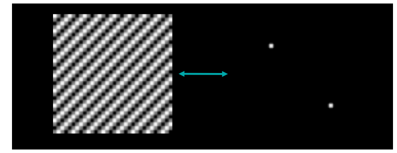
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2013
CT/Fourier Lecture 4

Examples



$$g(x,y) = \cos(2\pi(ax - by))$$

$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y + b) + \frac{1}{2}\delta(k_x + a)\delta(k_y - b)$$

Note: this is a corrected version of the example from Lecture 3.

Shift Theorem

$$F\{g(x-a)\} = G(k_x)e^{-j2\pi ak_x}$$

$$F[g(x-a,y-b)] = G(k_x,k_y)e^{-j2\pi(k_x a + k_y b)}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.
Each frequency component is shifted by a. This corresponds to a relative phase shift of

$$-2\pi a/(\text{spatial period}) \;=\; -2\pi a k_x$$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields
$\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$

Duality

Note the similarity between these two transforms

$$F\{e^{j2\pi ax}\} = \delta(k_x - a)$$
$$F\{\delta(x-a)\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\{G(x)\} = g(-k_x)$$

Application of Duality

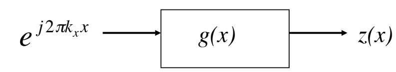
$$F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\{\Pi(x)\} = \text{sinc}(k_x)$.

Therefore from duality, $F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x)$

Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

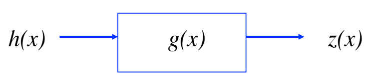


$$\begin{aligned} z(x) &= g(x) * e^{j2\pi k_x x} \\ &= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du \\ &= G(k_x) e^{j2\pi k_x x} \end{aligned}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

Convolution/Multiplication

Now consider an arbitrary input *h(x)*.



Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx \\ &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\ &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\ &= G(k_x) H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

Modulation

$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

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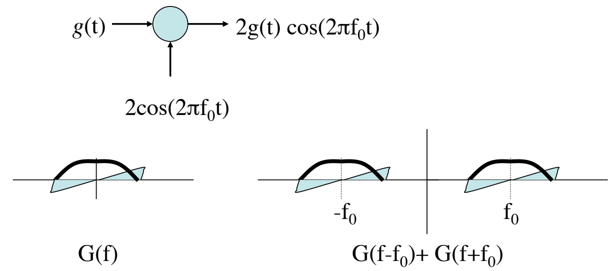
Example

Amplitude Modulation (e.g. AM Radio)



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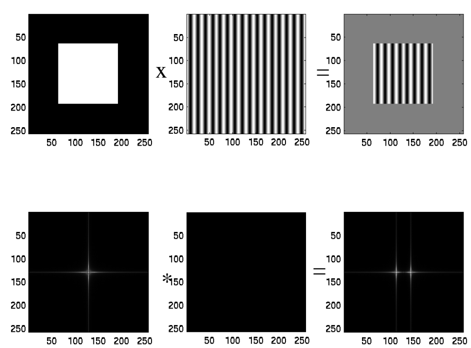
Modulation Example



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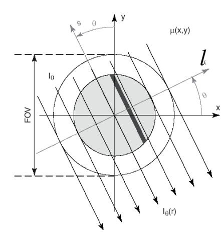
Radon Transform

$$g(l,\theta) = \int_{-\infty}^{\infty} \mu(x(s), y(s))\,ds$$
$$= \int_{-\infty}^{\infty} \mu(l\cos\theta - s\sin\theta, l\sin\theta + s\cos\theta)\,ds$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\delta\left(x\cos\theta + y\sin\theta - l\right)dx\,dy$$



$$\begin{bmatrix} l \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$l = x\cos\theta + y\sin\theta$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} l \\ s \end{bmatrix}$$

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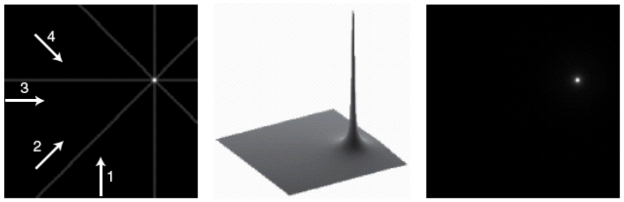
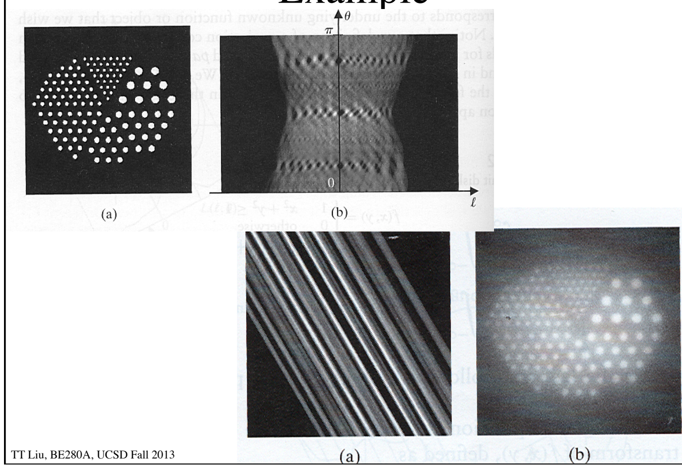
Backprojection



$$b(x,y) = B\left\{p(l,\theta)\right\} = \int_0^{\pi} p(x\cos\theta + y\sin\theta,\theta)\,d\theta$$

Suetens 2002

Example

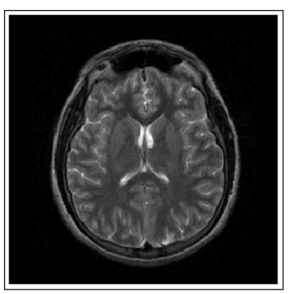


k-space

Image space

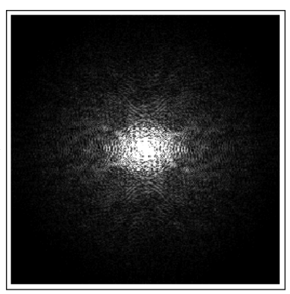
k-space

y



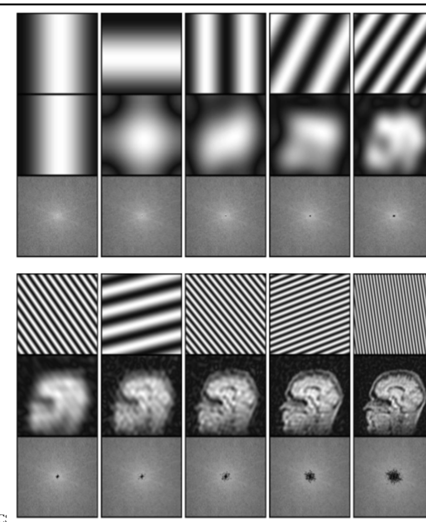
x

k_y



k_x

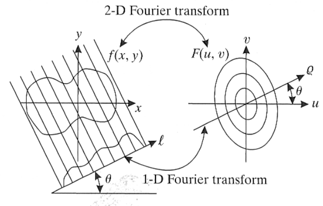
Fourier Transform



Hanson 2009

Projection-Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$



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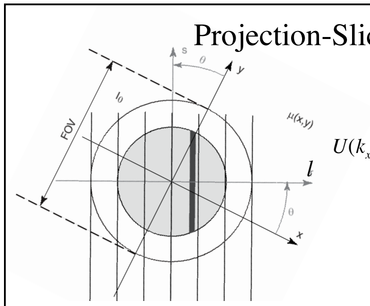
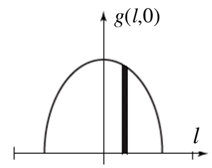
Projection-Slice Theorem



$$U(k_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dx dy$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \mu(x,y) dy\right] e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{\infty} g(x,0) e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{\infty} g(l,0) e^{-j2\pi k l} dl$$



In-Class Example:

$$\mu(x,y) = \cos 2\pi x$$

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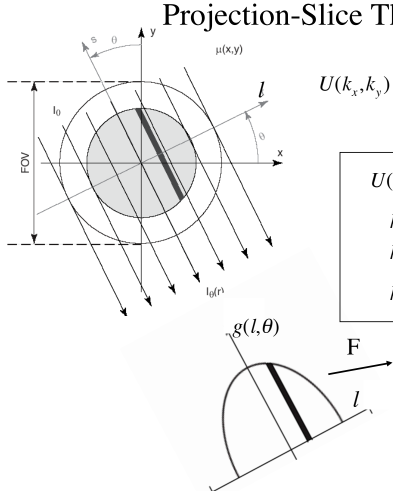
Projection-Slice Theorem



$$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dx dy$$
$$= F_{2D}\left[\mu(x,y)\right]$$

$$U(k_x,k_y) = G(k,\theta)$$
$$k_x = k\cos\theta$$
$$k_y = k\sin\theta$$
$$k = \sqrt{k_x^2 + k_y^2}$$

F

$$G(k,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi k l} dl$$

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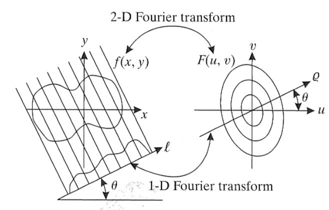
Projection-Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$



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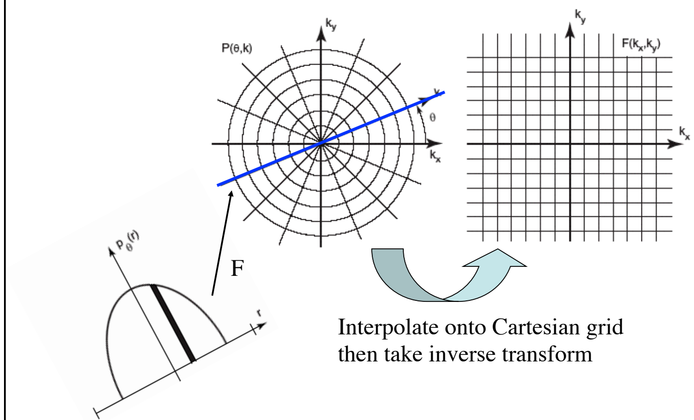
In-class Exercise

$$\mu(x,y) = rect(x,y)\cos\left(2\pi(x+y)\right)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

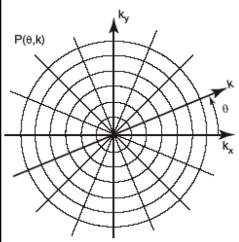
Fourier Reconstruction



Interpolate onto Cartesian grid
then take inverse transform

Suetens 2002

Polar Version of Inverse FT



$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x+k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta+yk\sin\theta)}k\,dk\,d\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta+y\sin\theta)}|k|\,dk\,d\theta$$

Note:

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

Suetens 2002

Filtered Backprojection

$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta+yk\sin\theta)}|k|\,dk\,d\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}\,dk\,d\theta$$
$$= \int_0^{\pi} g^*(l,\theta)\,d\theta$$

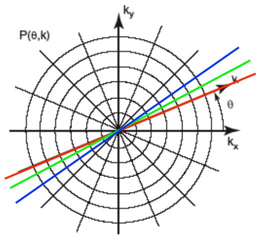
← Backproject a filtered projection

$where\ l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty}|k|G(k,\theta)e^{j2\pi kl}\,dk$$
$$= g(l,\theta) * F^{-1}\left[|k|\right]$$
$$= g(l,\theta) * q(l)$$

Suetens 2002

Fourier Interpretation



$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

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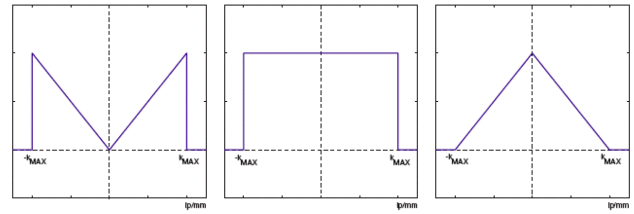
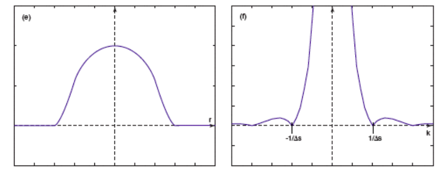
Ram-Lak Filter



$k_{max}=1/\Delta s$



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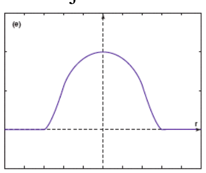
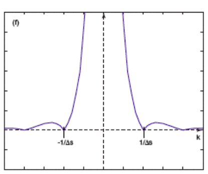
Suetens 2002

Reconstruction Path

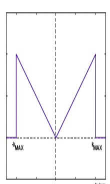
Projection



F



x



F^{-1} → Filtered Projection → Back-Project

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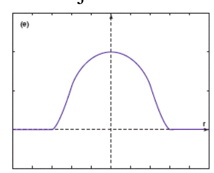
Reconstruction Path

Projection



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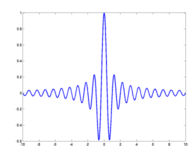

→ Filtered Projection

↓

Back-Project

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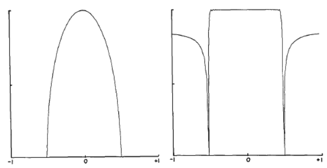
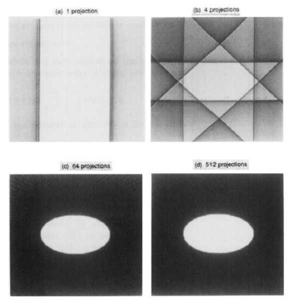
Suetens 2002

Example





Kak and Slaney

Example

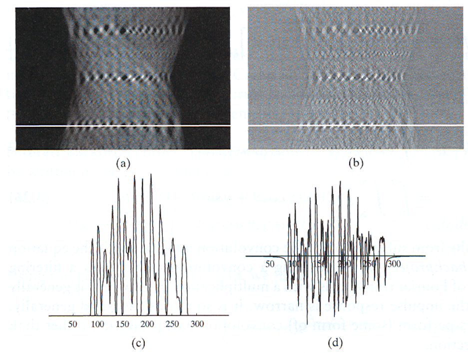


Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Prince and Links 2005

Example

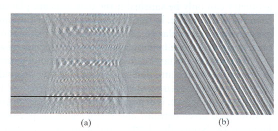


Figure 6.16
Backprojection step.

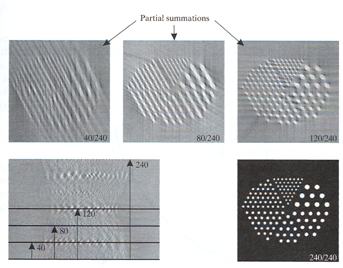


Figure 6.17
Summation step.

Prince and Links 2005