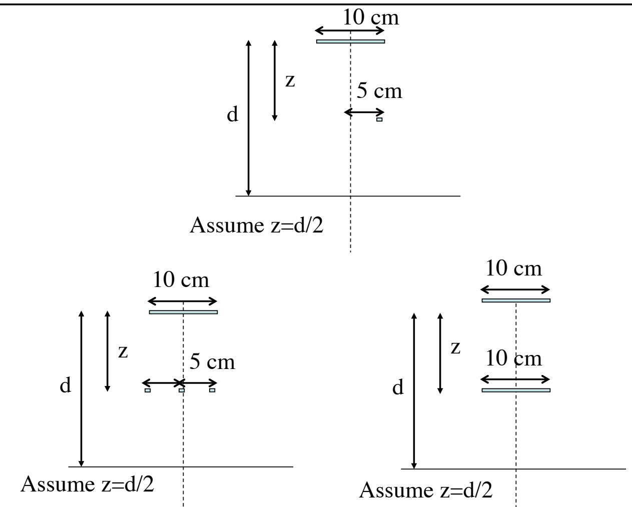
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
X-Rays Lecture 2



Convolution

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]] = h[m'-k]$$

$$\begin{aligned} y[m'] &= L\left[g[m]\right] \\ &= L\left[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\right] \\ &= L\left[g[0]\delta[m]\right] + L\left[g[1]\delta[m-1]\right] + L\left[g[2]\delta[m-2]\right] \\ &= g[0]L\left[\delta[m]\right] + g[1]L\left[\delta[m-1]\right] + g[2]L\left[\delta[m-2]\right] \\ &= g[0]h[m'-0] + g[1]h[m'-1] + g[2]h[m'-2] \\ &= \sum_{k=0}^{2} g[k]h[m'-k] \end{aligned}$$

1D Convolution

$$\begin{aligned} I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\ &= g(x) * h(x) \end{aligned}$$

Useful fact:

$$\begin{aligned} g(x) * \delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\ &= g(x-\Delta) \end{aligned}$$

2D Convolution

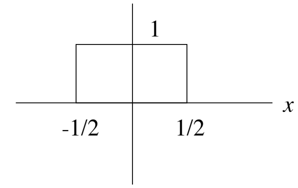
For a space invariant linear system, the superposition integral becomes a convolution integral.

$$
\begin{aligned}
I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\
&= g(x_2,y_2)**h(x_2,y_2)
\end{aligned}
$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2)= g(x_2, y_2)*h(x_2, y_2)$.

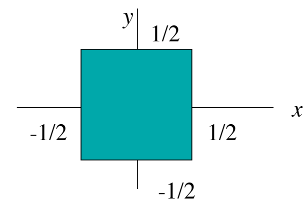
Rectangle Function

$$
\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}
$$

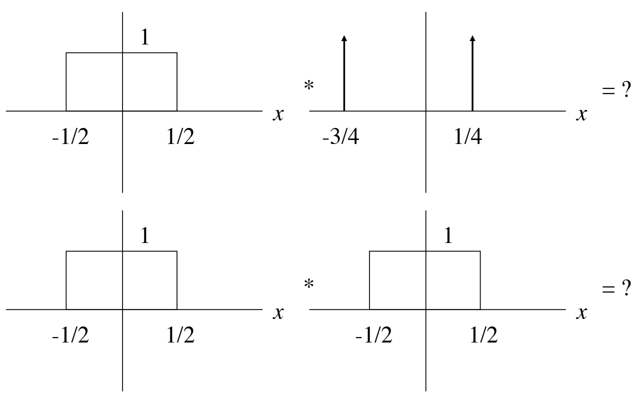


Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$

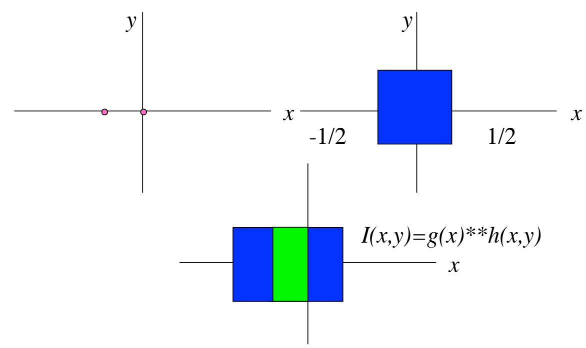


1D Convolution Examples

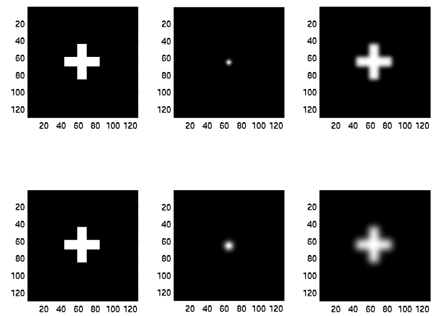


2D Convolution Example

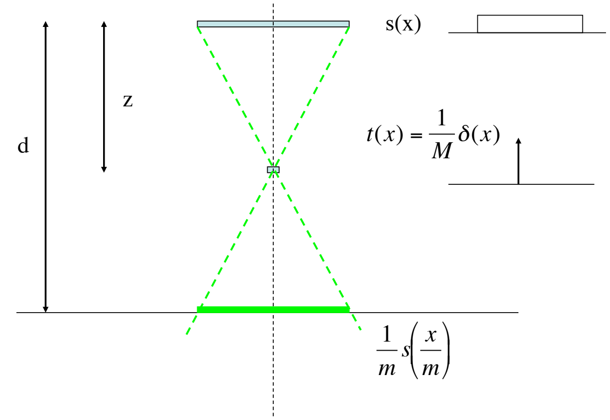
g(x)= δ(x+1/2,y) + δ(x,y) *h(x)=rect(x,y)*



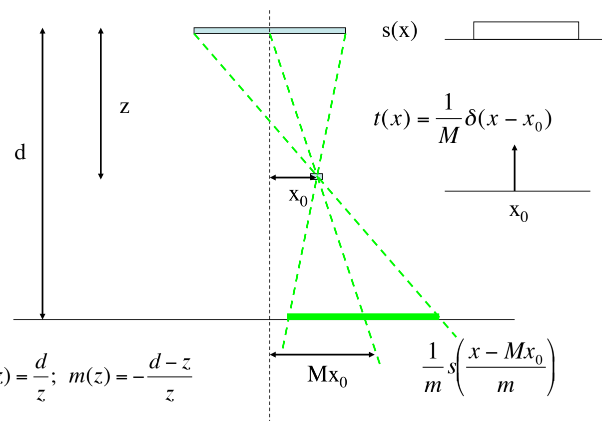
2D Convolution Example



X-Ray Imaging



X-Ray Imaging



$$M(z) = \frac{d}{z};\;\; m(z) = -\frac{d-z}{z}$$

X-Ray Imaging

For off-center pinhole object, the shifted source image can be written as

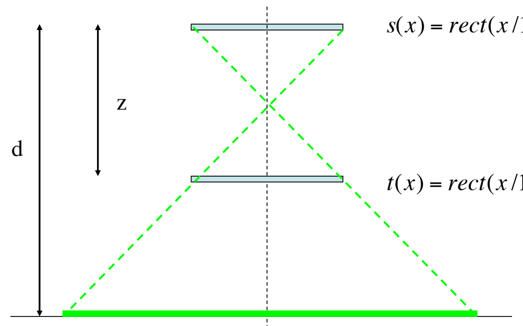
$$s\left(\frac{x - Mx_0}{m}\right) = s\left(\frac{x}{m}\right) * \frac{1}{M}\delta\left(\frac{x - Mx_0}{M}\right)$$
$$= s(x/m) * t\left(\frac{x}{M}\right)$$

For the general 2D case, we convolve the magnified object with the impulse response

$$I(x,y) = t\left(\frac{x}{M}, \frac{y}{M}\right) ** \frac{1}{m^2} s\left(\frac{x}{m}, \frac{y}{m}\right)$$

Note: we have ignored obliquity factors etc.

X-Ray Imaging



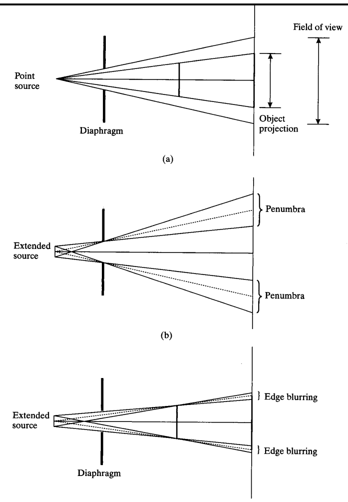
$M(z) = \frac{d}{z}; \;\; m(z) = -\frac{d-z}{z}$

$\frac{1}{m} s\left(\frac{x}{m}\right) * t\left(\frac{x}{M}\right)$

X-Ray Imaging

$m = 1; M = 2$

$$\frac{1}{m} s\left(\frac{x}{m}\right) * t\left(\frac{x}{M}\right) = rect(x/10) * rect(x/20)$$
$$= ???$$



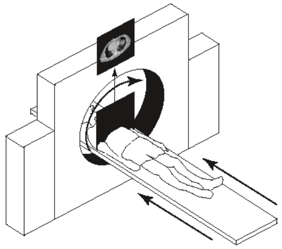
Prince and Link 2005

Figure 5.18
Effects of extended source. (a) Ideal field of view and object projection (with magnification). (b) Penumbra at edges of field of view due to extended source. (c) Blurred object edges due to extended source.

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.

Computed Tomography



http://www.youtube.com/watch?v=tqGmqRrxajQ

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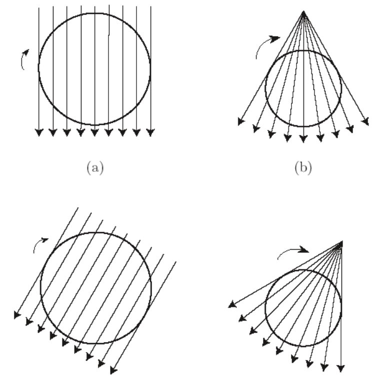
Computed Tomography

Parallel Beam



Fan Beam

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Scanner Generations

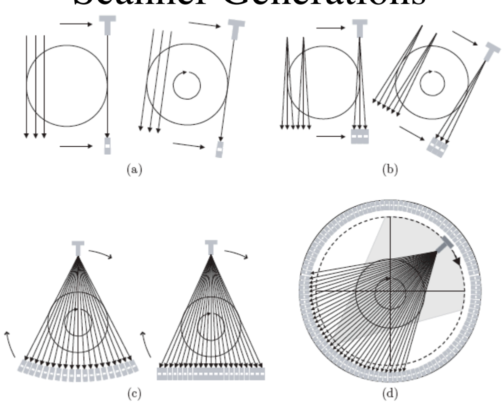

Figure 5.19: *Subsequent scanner generations: (a) first generation, (b) second generation, (c) third generation and (d) fourth generation CT scanner.*

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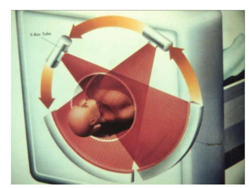
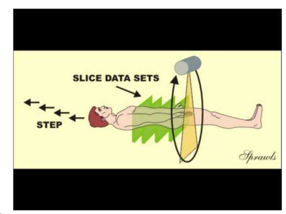
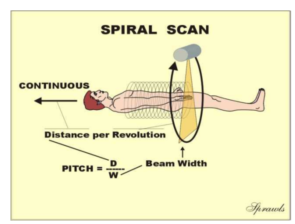
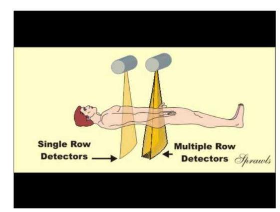


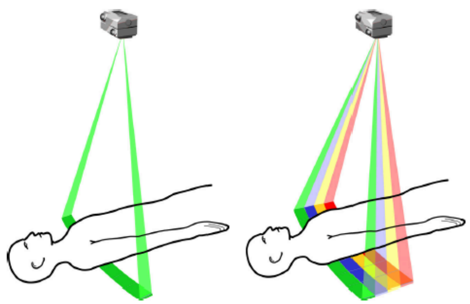






From http://www.sprawls.org/resources/CTIMG/classroom.htm

Single vs. Multi-slice



(a) (b)

Figure 5.22: *(a) Single-slice CT versus (b) multi-slice CT: a multi-slice CT scanner can acquire four slices simultaneously by using four adjacent detector arrays (Reprinted with permission of RSNA).*

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Scanner Generations

TABLE 6.1

Comparison of CT Generations

Generation	Source	Source Collimation	Detector	Detector Collimation	Source-Detector Movement	Advantages	Disadvantages
1G	Single x-ray tube	Pencil beam	Single	None	Move linearly and rotate in unison	Scattered energy is undetected	Slow
2G	Single x-ray tube	Fan beam, not enough to cover FOV	Multiple	Collimated to source direction	Move linearly and rotate in unison	Faster than 1G	Lower efficiency and larger noise because of the collimation in detectors
3G	Single x-ray tube	Fan beam, enough to cover FOV	Many	Collimated to source direction	Rotate in synchrony	Faster than 2G, continuous rotation using a slip ring	More expensive than 2G, low efficiency
4G	Single x-ray tube	Fan beam covers FOV	Stationary ring of detectors	Cannot collimate detectors	Detectors are fixed, source rotates	Higher efficiency than 3G	High scattering since detectors are not collimated
5G (EBCT)	Many tungsten anodes in single large tube	Fan beam	Stationary ring of detectors	Cannot collimate detectors	No moving parts	Extremely fast, capable of stop-action imaging of beating heart	High cost, difficult to calibrate
6G (Spiral CT)	3G/4G	3G/4G	3G/4G	3G/4G	3G/4G plus linear patient table motion	Fast 3D images	A bit more expensive
7G (Multislice CT)	Single x-ray tube	Cone beam	Multiple arrays of detectors	Collimated to source direction	3G/4G/6G motion	Fast 3D images	Expensive

Prince and Links 2005

1G vs. 2G scanner

Example 6.1 from Prince and Links.
Compare 1G vs. 2G scanner whose source - detector apparatus can move linearly at speed of 1 m/sec; FOV 0.5m; 360 projections over 180 degrees; 0.5 s for apparatus to rotate one angular increment, regardless of angle.
Question: Scan time for 1 G scanner? Scan time for 2G scanner with 9 detectors space 0.5 degrees apart?

Answer:

1G scanner: 0.5m/(1m/s) = 0.5s per projection.
360 * 0.5 = 180s scan time
360 * 0.5 = 180s for rotation of apparatus.
Total time = 360 s or 6 minutes.

2G scanner: Required angular resolution is 180/360 = 0.5 degrees -- agrees with spacing.
360/9 = 40 rotations required.
40 * 0.5 = 20s for scanning
40 * 0.5 = 20s for rotations.
Total time = 40s.

3G, 6G, and 7G scanners

3G scanner: Typical scanner acquires 1000 projections with fanbeam angle of 30 to 60 degrees; 500 to 700 detectors; 1 to 20 seconds.

6G: Spiral/Helical CT
60 cm torso scan: 30s.
24 cm lung scan: 12s
15 cm angio: 30s

7G: Multislice CT
64 or more parallel 1D projections.

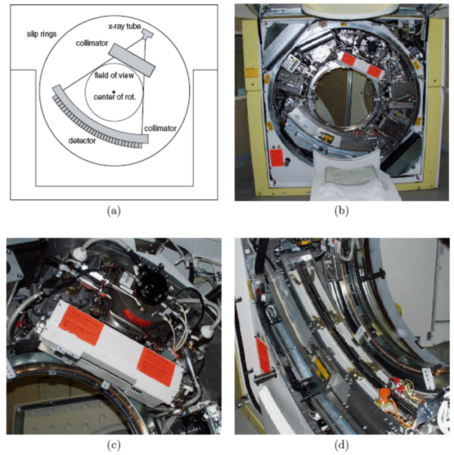

Figure 5.20: *(a-b) The basic internal geometry of a third generation spiral CT scanner. (c) X-ray tube with adjustable collimating split. (d) Detector array with post-patient collimator.*

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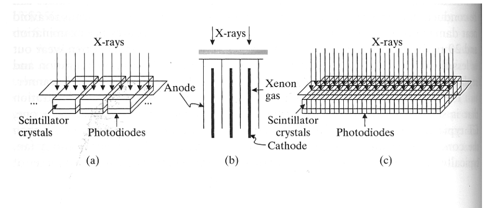
Detectors

Figure 6.7
(a) Solid-state detectors,
(b) xenon gas detectors, and
(c) multiple (solid-state) detector array.



Prince and Links 2005

CT Line Integral

$$I_d = \int_0^{E_{\max}} S_0(E) E \exp\left(-\int_0^d \mu(s;E')ds\right) dE$$

Monoenergetic Approximation

$$I_d = I_0 \exp\left(-\int_0^d \mu(s;\overline{E})ds\right)$$

$$g_d = -\log\left(\frac{I_d}{I_0}\right) = \int_0^d \mu(s;\overline{E})ds$$

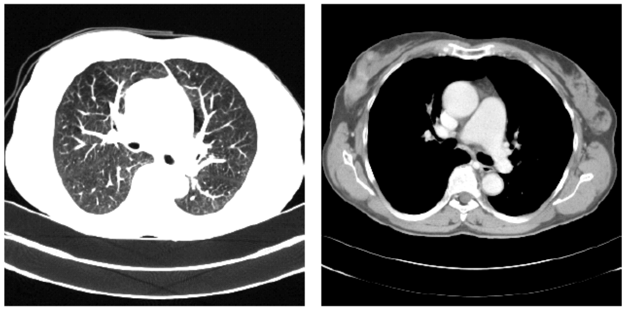
CT Number

$$\text{CT\_number} = \frac{\mu - \mu_{water}}{\mu_{water}} \times 1000$$

Measured in Hounsfield Units (HU)

Air: -1000 HU
Soft Tissue: -100 to 60 HU
Cortical Bones: 250 to 1000 HU
Metal and Contrast Agents: > 2000 HU

CT Display



(a) (b)

Figure 5.4: *CT-image of the chest with different window/level settings:(a) for the lungs (window 1500 and level -500) and (b) for the soft tissues (window 350 and level 50).*

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Direct Inverse Approach

μ_1	μ_2	p_1
μ_3	μ_4	p_2
p_3	p_4	

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_4 = \mu_2 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns.
Are these the correct equations to use?

Direct Inverse Approach

μ_1	μ_2	p_1
μ_3	μ_4	p_2
p_3	p_4	

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_4 = \mu_2 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns.
Are these the correct equations to use?

Direct Inverse Approach

μ_1	μ_2	p_1
μ_3	μ_4	p_2
p_3	p_4	p_5

$p_1 = \mu_1 + \mu_2$
$p_2 = \mu_3 + \mu_4$
$p_3 = \mu_1 + \mu_3$
$p_5 = \mu_1 + \mu_4$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

4 equations, 4 unknowns. These are linearly independent now.
In general for a NxN image, N^2 unknowns, N^2 equations.
This requires the inversion of a $N^2 x N^2$ matrix
For a high-resolution 512x512 image, $N^2=262144$ equations.
Requires inversion of a 262144x262144 matrix!
Inversion process sensitive to measurement errors.

Iterative Inverse Approach
Algebraic Reconstruction Technique (ART)

1	2	3
3	4	7
4	6	5

→

2.5	2.5	5
2.5	2.5	5

↓

1.5	1.5	3
3.5	3.5	7
5	5	

←

1	2	3
3	4	7
5	5	

Backprojection

0	0	0	0
0	3	0	3
0	0	0	0
0	3	0	

Diagonal sums: 3, 3

0	0	0
1	1	1
0	0	0

→

1	0	0
1	2	1
0	0	1

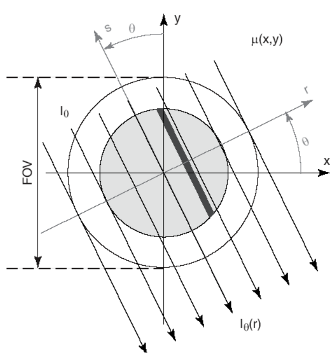
→

1	1	0
1	3	1
0	1	1

→

1	1	1
1	4	1
1	1	1

Projections



$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

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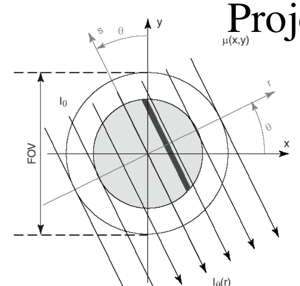
Projections



$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(x,y)ds\right)$$
$$= I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

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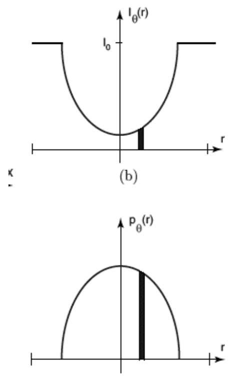
Projections



$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

$$p(r,\theta) = -\ln\frac{I_\theta(r)}{I_0}$$

$$= \int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

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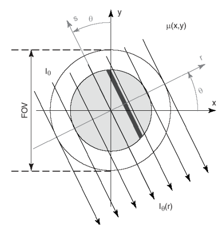
Radon Transform

$$g(r,\theta) = \int_{-\infty}^{\infty} \mu(x(s),y(s))ds$$

$$= \int_{-\infty}^{\infty} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\delta(x\cos\theta + y\sin\theta - r)dxdy$$



$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad r = x\cos\theta + y\sin\theta$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

Example

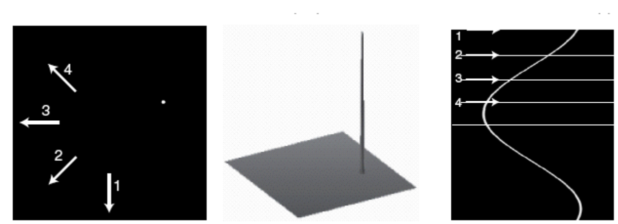
$$f(x,y) = \begin{cases} 1 & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(l,\theta = 0) = \int_{-\infty}^{\infty} f(l,y)dy$$

$$= \int_{-\sqrt{1-l^2}}^{\sqrt{1-l^2}} dy$$

$$= \begin{cases} 2\sqrt{1-l^2} & |l| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Sinogram



(a) (b) (c)

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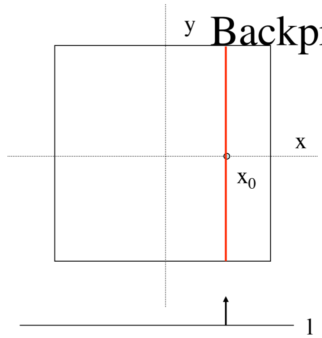
Backprojection



$$b(x_0, y) = p(l, \theta = 0)\Delta\theta$$
$$= p(x_0, 0)\Delta\theta$$

$$b_\theta(x, y) = g(x\cos\theta + y\sin\theta, \theta)\Delta\theta$$
$$b(x, y) = B\{g(l, \theta)\}$$
$$= \int_0^\pi g(x\cos\theta + y\sin\theta, \theta)d\theta$$

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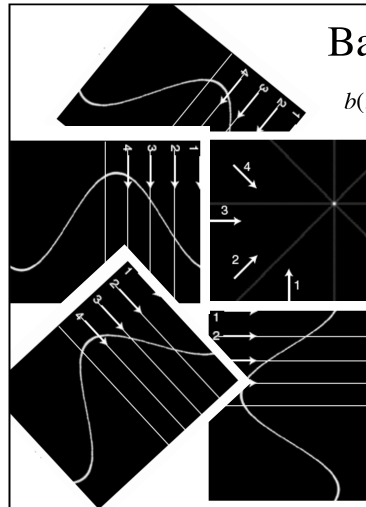
Backprojection



$$b(x, y) = B\{p(l, \theta)\}$$
$$= \int_0^\pi p(x\cos\theta + y\sin\theta, \theta)d\theta$$

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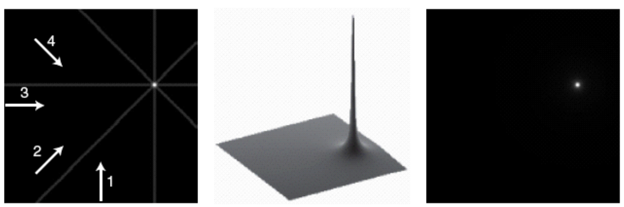
Backprojection



$$b(x, y) = B\{p(l, \theta)\} = \int_0^\pi p(x\cos\theta + y\sin\theta, \theta)d\theta$$

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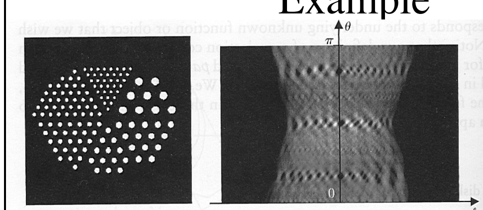
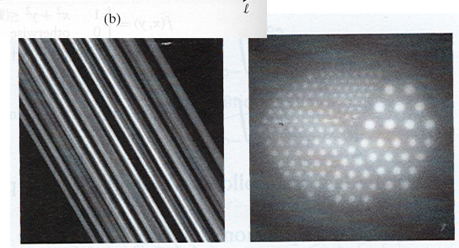
Example



(a) (b)


(a) (b)