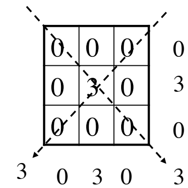
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
CT/Fourier Lecture 2

Backprojection



0	0	0
1	1	1
0	0	0

→

1	0	0
1	2	1
0	0	1

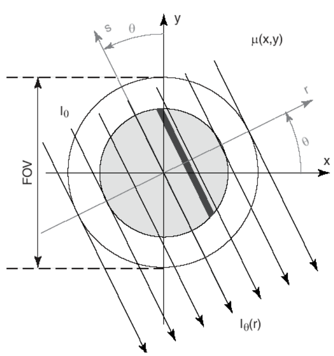
→

1	1	0
1	3	1
0	1	1

→

1	1	1
1	4	1
1	1	1

Projections

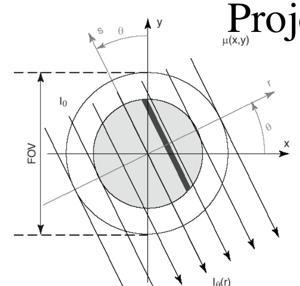


$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

Suetens 2002

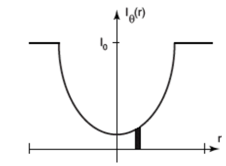
Projections



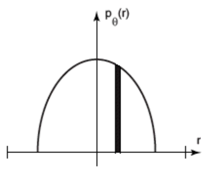
$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(x,y)ds\right)$$
$$= I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

Suetens 2002

Projections



(b)



(c)

$$I(r,\theta) = I_0 \exp\left(-\int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds\right)$$

$$p(r,\theta) = -\ln\frac{I_\theta(r)}{I_0}$$

$$= \int_{L_{r,\theta}} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

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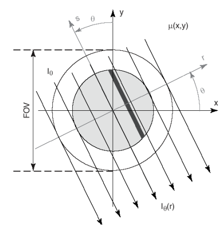
Radon Transform

$$g(r,\theta) = \int_{-\infty}^{\infty} \mu(x(s), y(s))ds$$

$$= \int_{-\infty}^{\infty} \mu(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\delta\left(x\cos\theta + y\sin\theta - r\right)dxdy$$



$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad r = x\cos\theta + y\sin\theta$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

Example

$$f(x,y) = \begin{cases} 1 & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(l,\theta = 0) = \int_{-\infty}^{\infty} f(l,y)dy$$

$$= \int_{-\sqrt{1-l^2}}^{\sqrt{1-l^2}} dy$$

$$= \begin{cases} 2\sqrt{1-l^2} & |l| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example

$$f(x,y) = rect(x,y)$$

Calculate the projections at angles of 0, 45, and 90 degrees

Example

$$f(x,y) = \delta(x-10, y-10)$$

What is the projection of this object?

In-class Exercise

$$\mu(x,y) = rect(x, y/2)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

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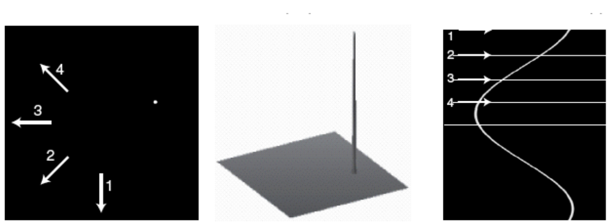
Sinogram



(a) (b) (c)

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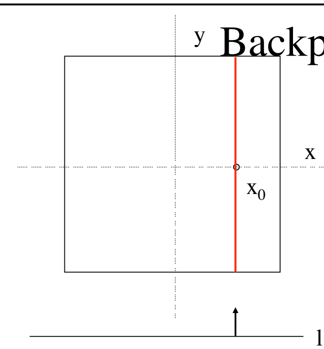
Backprojection



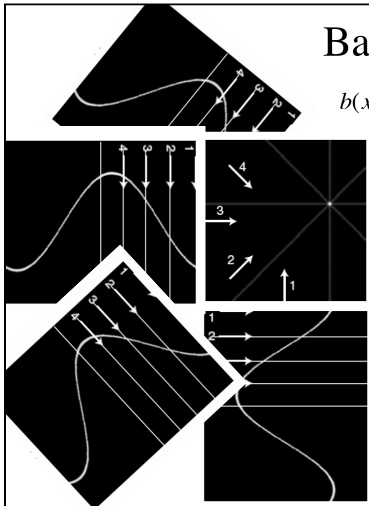
$$b(x_0, y) = p(l, \theta = 0)\Delta\theta$$
$$= p(x_0, 0)\Delta\theta$$

$$b_\theta(x,y) = g(x\cos\theta + y\sin\theta, \theta)\Delta\theta$$

$$b(x,y) = B\{g(l,\theta)\}$$
$$= \int_0^\pi g(x\cos\theta + y\sin\theta, \theta)d\theta$$

Suetens 2002

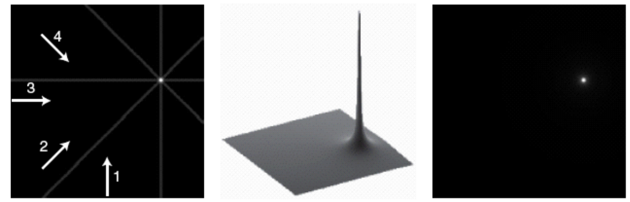
Backprojection



$$b(x,y) = B\{p(l,\theta)\}$$
$$= \int_0^\pi p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$

Suetens 2002

Backprojection



$$b(x,y) = B\{p(l,\theta)\} = \int_0^\pi p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$

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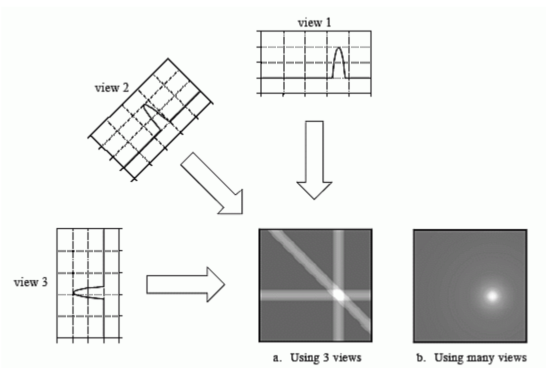

FIGURE 25-16
Backprojection. Backprojection reconstructs an image by taking each view and *smearing* it along the path it was originally acquired. The resulting image is a blurry version of the correct image.

http://www.dspguide.com/ch25/5.htm

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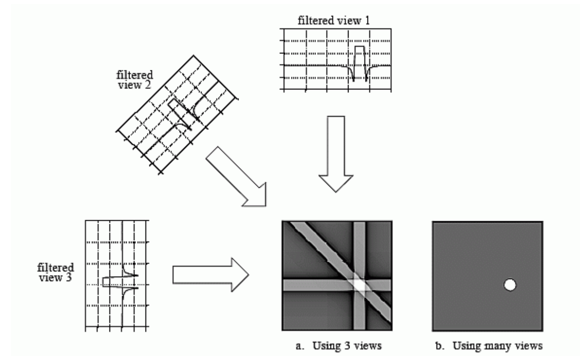

FIGURE 25-17
Filtered backprojection. Filtered backprojection reconstructs an image by filtering each view before backprojection. This removes the blurring seen in simple backprojection, and results in a mathematically exact reconstruction of the image. Filtered backprojection is the most commonly used algorithm for computed tomography systems.

http://www.dspguide.com/ch25/5.htm

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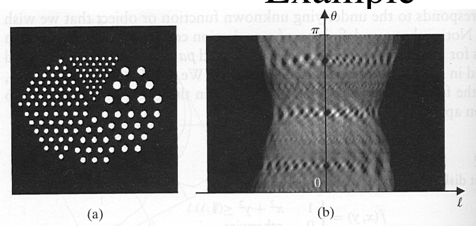
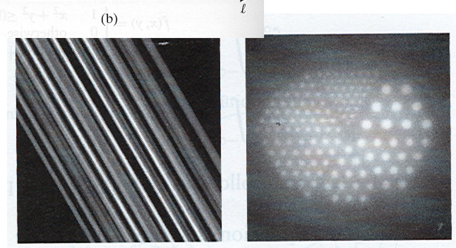
Example





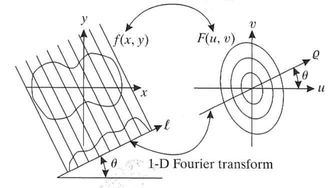
Projection Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$

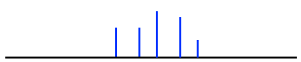
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$

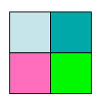


Prince&Links 2006

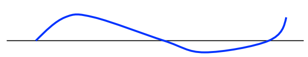
Signals and Images

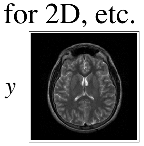
Discrete-time/space signal /image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.





Continuous-time/space signal / image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.





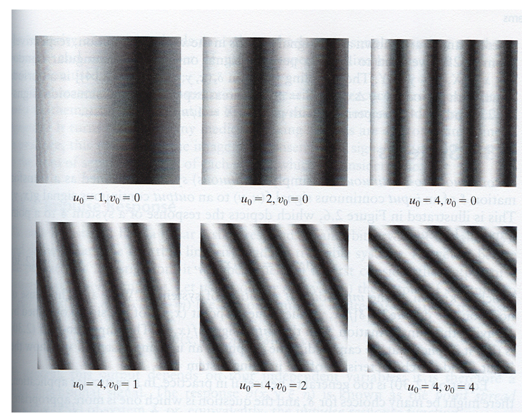
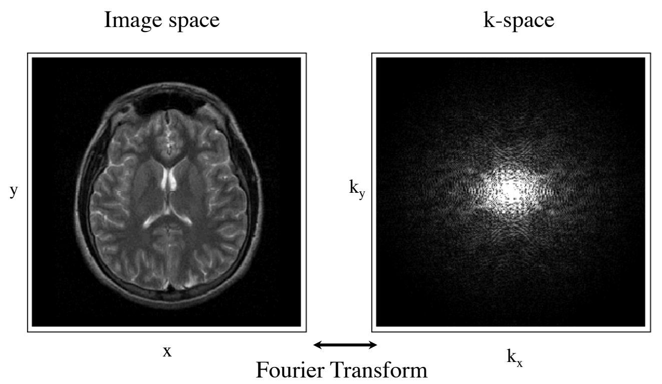


Figure 2.5 from Prince and Link

k-space



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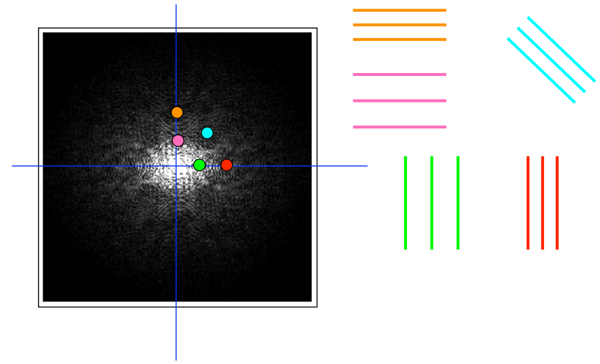

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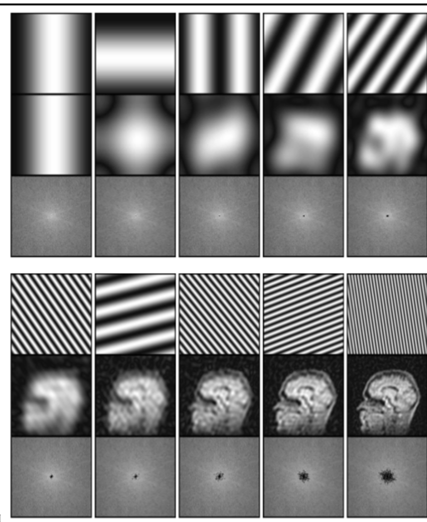

Hanson 2009

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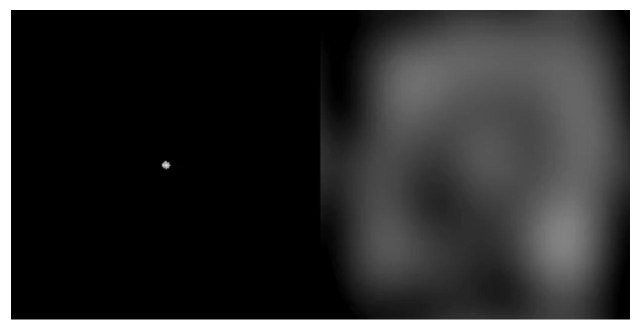

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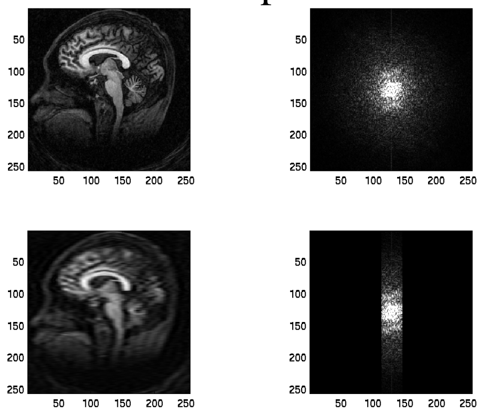
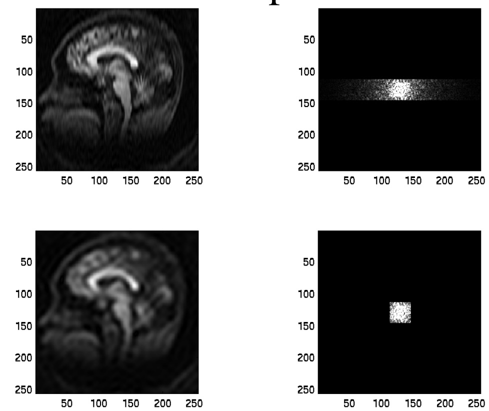
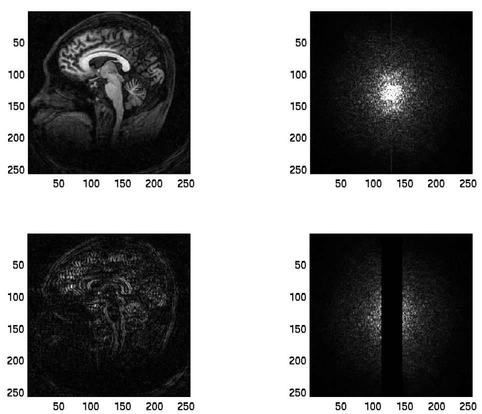
Examples



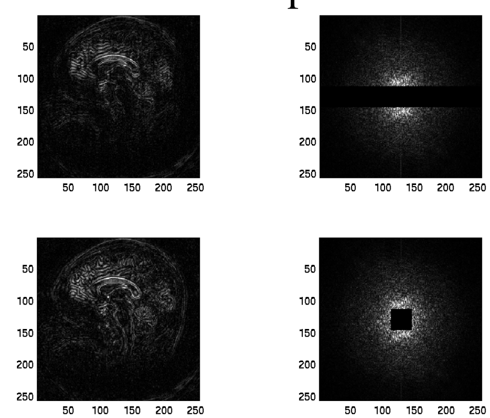
Examples



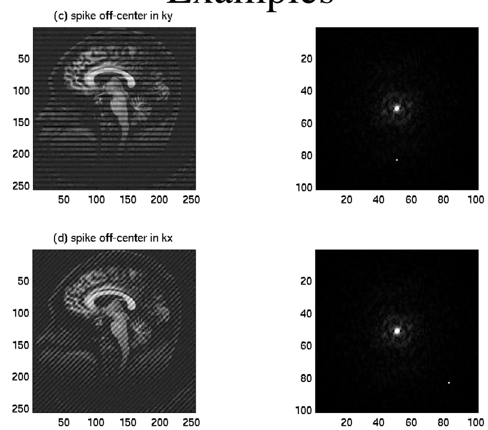
Examples



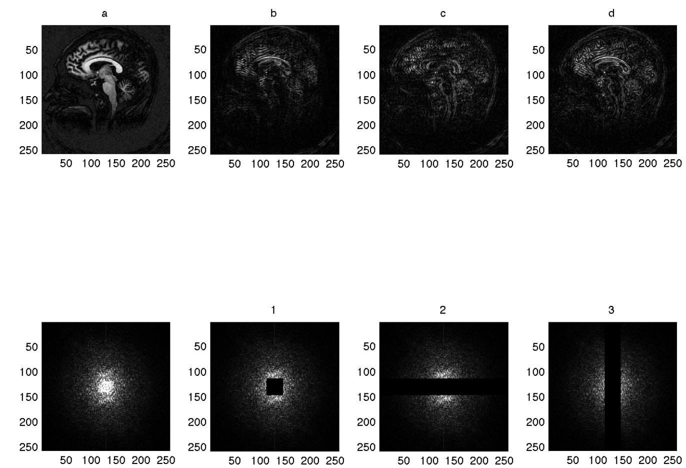
Examples



Examples



TT Li



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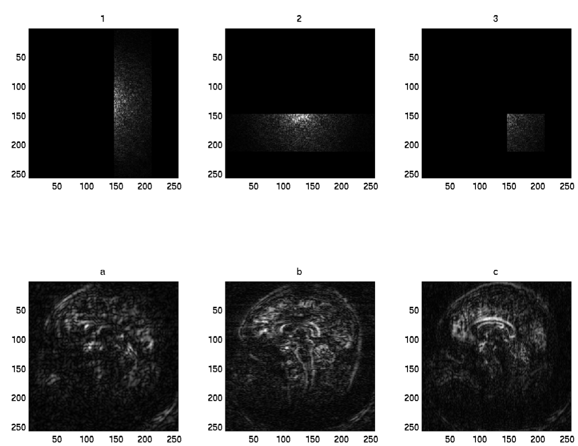

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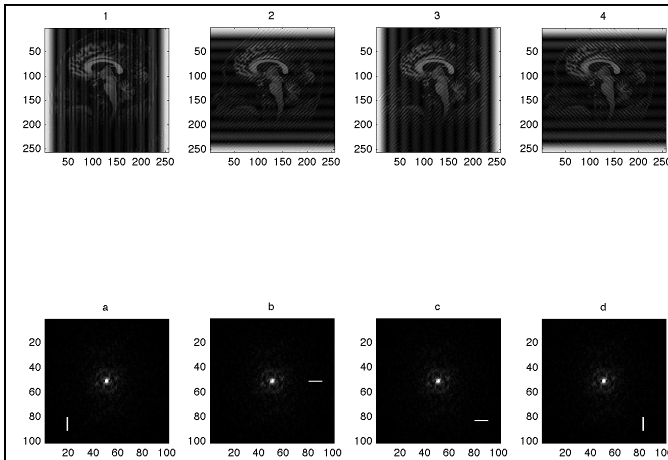

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The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$ Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$ Inverse Fourier Transform

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi k_x x} dx$$ Fourier Transform

$$g(x) = \int_{-\infty}^{\infty} G(k_x) e^{j2\pi k_x x} dk_x$$ Inverse Fourier Transform

Complex Numbers

$$j = \sqrt{-1}$$

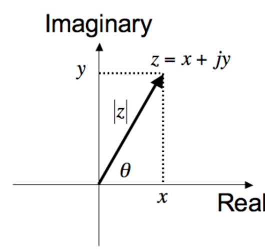
$$j^2 = ?$$

$$(3+2j)(3-2j) = ?$$

$$j^2 = -1$$

$$(3+2j)(3-2j) = 9 - 4j^2 = 13$$

Complex Numbers



$$z = 2 + 1j$$

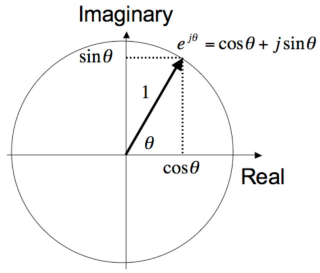
$$|z| = \sqrt{2^2 + 1} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.6 \text{ degrees}$$

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy = |z|e^{j\theta}$$



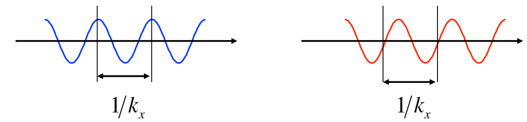
1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$



Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi k_x x}dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0)e^{-j2\pi k_x x}dx = e^{-j2\pi k_x x_0}$$

$$F(\Pi(x)) = \int_{-1/2}^{1/2} e^{-j2\pi k_x x}dx$$

$$= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x}$$

$$= \frac{\sin(\pi k_x)}{\pi k_x} = \operatorname{sinc}(k_x)$$

2D Fourier Transform

Fourier Transform

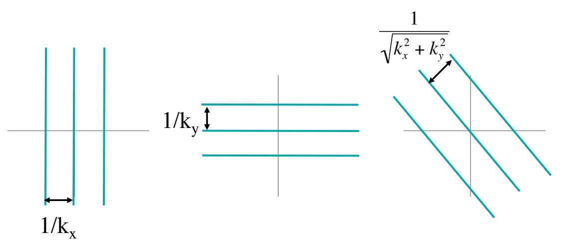
$$G(k_x, k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x, k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$

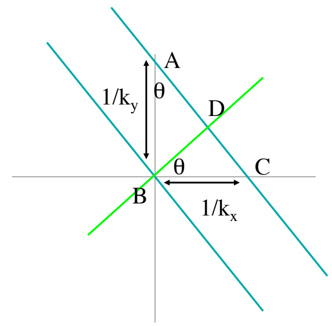
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

Plane Waves



$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2} + \frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$