**Bioengineering 280A**  
Principles of Biomedical Imaging  
Fall Quarter 2014  
CT/Fourier Lecture 3

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**k-space**

Image space

- k-space
- Fourier Transform

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Hanson 2009
The Fourier Transform

Fourier Transform (FT)

\[ G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt = F\{g(t)\} \]

Inverse Fourier Transform

\[ g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df = F^{-1}\{G(f)\} \]

1D Fourier Transform

\[ G(k_x) = \int_{-\infty}^{\infty} g(x) \exp(-j2\pi k_x x) \, dx \]

\[ = \int_{-\infty}^{\infty} g(x) \cos(2\pi k_x x) \, dx - j \int_{-\infty}^{\infty} g(x) \sin(2\pi k_x x) \, dx \]

The part of \(g(x)\) that "looks" like \(\cos(2\pi k_x x)\)

The part of \(g(x)\) that "looks" like \(\sin(2\pi k_x x)\)

Computing Transforms

\[ F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} \, dx = 1 \]

\[ F(\delta(x-x_0)) = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-j2\pi k_x x} \, dx = e^{-j2\pi k_x x_0} \]

\[ F(\Pi(x)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi k_x x} \, dx = 2 \sin(\pi k_x) \pi k_x \]

\[ = \frac{\sin(\pi k_x)}{\pi k_x} = \text{sinc}(k_x) \]

Computing Transforms

\[ F(I) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx = ??? \]

Define \(h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx\) and see what it does under an integral.

\[ \int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x = \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx \, dk_x \]

\[ = \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} \, dx \, dk_x \]

\[ = \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} \, dx \, dk_x \]

\[ = G(0) \]

Therefore, \(F(I) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx = \delta(k_x)\)
Computing Transforms

Similarly,

\[ F\{e^{j2\pi k_0 x}\} = \delta(k_x - k_0) \]
\[ F\{\cos 2\pi k_0 x\} = \frac{1}{2}(\delta(k_x - k_0) + \delta(k_x + k_0)) \]
\[ F\{\sin 2\pi k_0 x\} = \frac{1}{2j}(\delta(k_x - k_0) - \delta(k_x + k_0)) \]

2D Fourier Transform

Fourier Transform

\[ G(k_x, k_y) = F\{g(x, y)\} = \frac{1}{\sqrt{2\pi^2 k_x^2 + k_y^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(k_x x + k_y y)} \, dx \, dy \]

Inverse Fourier Transform

\[ g(x, y) = \frac{1}{\sqrt{2\pi^2 k_x^2 + k_y^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} \, dk_x \, dk_y \]

Plane Waves

\[ e^{j2\pi(k_x x + k_y y)} = \cos(2\pi(k_x x + k_y y)) + j\sin(2\pi(k_x x + k_y y)) \]

Figure 2.5 from Prince and Link
Plane Waves

$\Delta ABC \sim \Delta BDC$

$\frac{AC}{AB} \frac{BD}{BC} = -\frac{1}{k_x} \frac{1}{k_y} \frac{1}{\sqrt{k_x^2 + k_y^2}} = \frac{1}{k_x} \frac{1}{k_y}$

$\theta = \arctan\left(\frac{k_y}{k_x}\right)$

Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_x(x)g_y(y)$

The Fourier Transform is then separable as well.

$G(k_x,k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)e^{-j2\pi (k_xx + k_yy)} dxdy$

$= \int_{-\infty}^{\infty} g_x(x)e^{-j2\pi k_xx} dx \int_{-\infty}^{\infty} g_y(y)e^{-j2\pi k_yy} dy$

$= G_x(k_x)G_y(k_y)$

Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$

Example (sinc/rect)

Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$

Examples

Is this function separable?

$g(x,y) = \exp(-j2\pi(8x + 9y))\sin(28\pi x)$

PollEv.com/be280a
**Examples**

\[ g(x, y) = \delta(x, y) = \delta(x)\delta(y) \]
\[ G(k_x, k_y) = 1 \]

\[ g(x, y) = \delta(x) \]
\[ G(k_x, k_y) = \delta(k_x) \]

**Linearity**

The Fourier Transform is linear.

\[ F\{ag(x) + bh(x)\} = aG(k_x) + bH(k_y) \]

**Examples**

\[ g(x, y) = 1 + e^{-j2\pi ax} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x + a)\delta(k_y) \]

\[ g(x, y) = 1 + e^{j2\pi bx} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x - a)\delta(k_y) \]

\[ g(x, y) = \cos(2\pi(ax - by)) \]
\[ G(k_x, k_y) = \frac{1}{2} \delta(k_x - a)\delta(k_y + b) + \frac{1}{2} \delta(k_x + a)\delta(k_y - b) \]
Scaling Theorem

\[ F\{g(ax)\} = \frac{1}{|a|} \delta \left( \frac{k_x}{a} \right) \]

\[ F\{g(ax, by)\} = \frac{1}{|ab|} \delta \left( \frac{k_x}{a}, \frac{k_y}{b} \right) \]

MTF = Fourier Transform of PSF

Modulation Transfer Function (MTF)

or

Frequency Response

Bushberg et al. 2001
Duality

Note the similarity between these two transforms

\[
\begin{align*}
F\{e^{j2\pi ax}\} &= \delta(kx-a) \\
F\{\delta(x-a)\} &= e^{-j2\pi kx} 
\end{align*}
\]

These are specific cases of duality

\[
F\{G(x)\} = g(-k_x)
\]

Application of Duality

Recall that \( F\{\Pi(x)\} = \text{sinc}(k_x) \).

Therefore from duality, \( F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x) \)
Shift Theorem

\[ F\{g(x-a)\} = G(k)e^{-j2\pi ak} \]

\[ F\{g(x-a,y-b)\} = G(k_x,k_y)e^{-j2\pi(k_xa+k_yb)} \]

Shifting the function doesn’t change its spectral content, so the magnitude of the transform is unchanged. Each frequency component is shifted by \(a\). This corresponds to a relative phase shift of

\[-2\pi a / \text{(spatial period)} = -2\pi ak_x \]

For example, consider \(\exp(j2\pi k_x x)\). Shifting this by \(a\) yields \(\exp(j2\pi k_x (x-a)) = \exp(\exp(j2\pi k_x x) \exp(-j2\pi ak_x)\)

Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

\[ e^{j2\pi k_x x} \rightarrow g(x) \rightarrow z(x) \]

\[ z(x) = g(x) * e^{j2\pi k_x x} \]

\[ = \int_{-\infty}^{\infty} g(u)e^{j2\pi k_x (x-u)} du \]

\[ = G(k_x)e^{j2\pi k_x x} \]

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

Convolution/Multiplication Theorem

Now consider an arbitrary input \(h(x)\).

\[ h(x) \rightarrow g(x) \rightarrow z(x) \]

Recall that we can express \(h(x)\) as the integral of weighted complex exponentials.

\[ h(x) = \int_{-\infty}^{\infty} H(k)e^{j2\pi k_x x} dk_x \]

Each of these exponentials is weighted by \(G(k)\) so that the response may be written as

\[ z(x) = \int_{-\infty}^{\infty} G(k_x)H(k_x)e^{j2\pi k_x x} dk_x \]

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

\[ F\{g(x)h(x)\} = G(k_x)H(k_x) \]
2D Convolution/Multiplication

Convolution
\[ F[g(x,y) \ast h(x,y)] = G(k_x,k_y)H(k_x,k_y) \]

Multiplication
\[ F[g(x,y)h(x,y)] = G(k_x,k_y) \ast H(k_x,k_y) \]

Application of Convolution Thm.
\[ \Lambda(x) = \begin{cases} 
1 - |x| & |x| < 1 \\
0 & \text{otherwise}
\end{cases} \]
\[ F(\Lambda(x)) = \int_{-1}^{1} (1 - |x|) e^{-j2\pi k_x x} \, dx = ?? \]

Application of Convolution Thm.
\[ \Lambda(x) = \Pi(x) \ast \Pi(x) \]
\[ F(\Lambda(x)) = \sin^2(k_x) \]

Convolution Example
Response of an Imaging System

\[ g(x,y) \xrightarrow{H_1} h_1(x,y) \xrightarrow{H_2} h_2(x,y) \xrightarrow{H_3} z(x,y) \]

\[ Z(k_x,k_y) = G(k_x,k_y) H_1(k_x,k_y) H_2(k_x,k_y) H_3(k_x,k_y) \]

System MTF = Product of MTFs of Components

Useful Approximation

\[ FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots + FWHM_N^2} \]

Example

- FWHM_1 = 1mm
- FWHM_2 = 2mm
- FWHM_{system} = \sqrt{5} = 2.24mm

**D14.** The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is _____ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5
Modulation

\[ F[g(x)e^{j2\pi f_0 x}] = G(k_x - k_0) \]

\[ F[g(x)\cos(2\pi f_0 x)] = \frac{1}{2} G(k_x - k_0) + \frac{1}{2} G(k_x + k_0) \]

\[ F[g(x)\sin(2\pi f_0 x)] = \frac{1}{2j} G(k_x - k_0) - \frac{1}{2j} G(k_x + k_0) \]

Example

Amplitude Modulation (e.g. AM Radio)

\[ g(t) \rightarrow 2g(t) \cos(2\pi f_0 t) \]

Summary of Basic Properties

- **Linearity**
  \[ F[a g(x, y) + b h(x, y)] = a G(k_x, k_y) + b H(k_x, k_y) \]

- **Scaling**
  \[ F[g(ax, by)] = \frac{1}{ab} F\left(\frac{k_x}{a}, \frac{k_y}{b}\right) \]

- **Duality**
  \[ F[G(x)] = g(-k)_x \]

- **Shift**
  \[ F[g(x-a, y-b)] = G(k_x, k_y)e^{-j2\pi(k_x a + k_y b)} \]

- **Convolution**
  \[ F[g(x) * h(x)] = G(k_x)H(k_x) \]

- **Multiplication**
  \[ F[g(x)h(x)] = G(k_x) * H(k_x) \]

- **Modulation**
  \[ F[g(x)e^{j2\pi f_0 x}] = G(k_x - f_0) \]