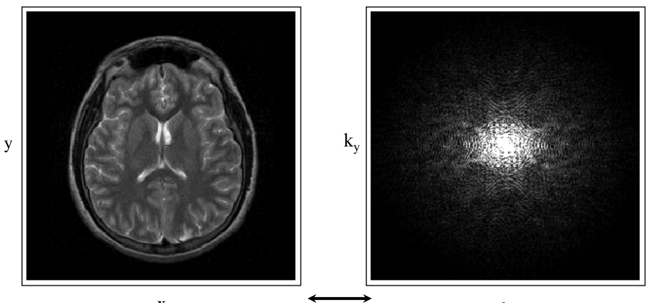
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
CT/Fourier Lecture 3

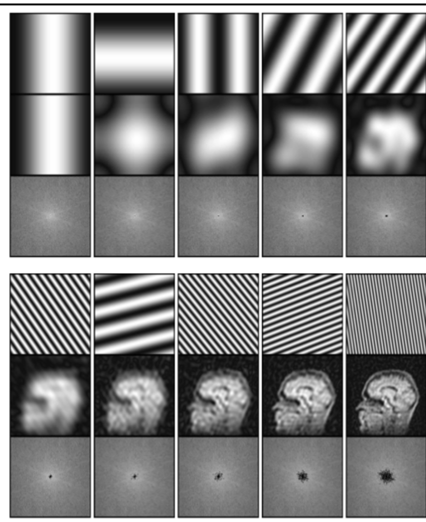
k-space

Image space k-space

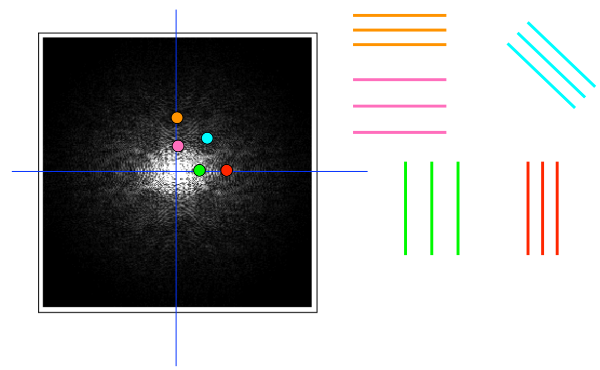
y k_y



x Fourier Transform k_x



Hanson 2009



The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df = F^{-1}\{G(f)\}$$

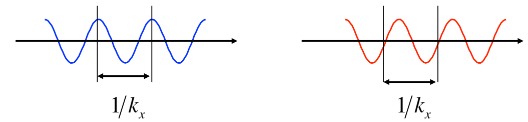
1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$



Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$F(\Pi(x)) = \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx$$

$$= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x}$$

$$= \frac{\sin(\pi k_x)}{\pi k_x} = \mathrm{sin}c(k_x)$$

Computing Transforms

$$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x = \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx$$

$$= \int_{-\infty}^{\infty} g(-x) dx$$

$$= G(0)$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

Computing Transforms

Similarly,

$$F\left\{e^{j2\pi k_0 x}\right\} = \delta(k_x - k_0)$$

$$F\{\cos 2\pi k_0 x\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$

$$F\{\sin 2\pi k_0 x\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$

2D Fourier Transform

Fourier Transform

$$G(k_x, k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi\left(k_x x + k_y y\right)} dx dy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi\left(k_x x + k_y y\right)} dk_x dk_y$$

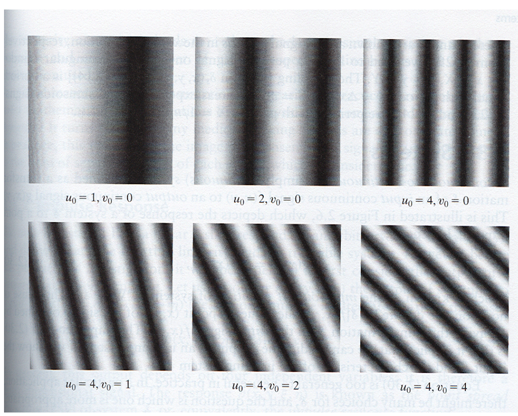
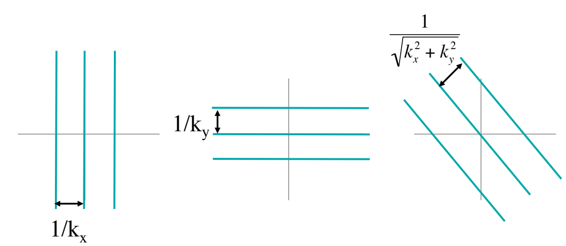


Figure 2.5 from Prince and Link

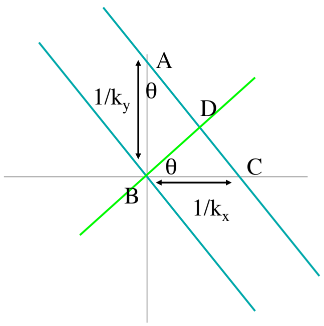
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

Plane Waves



$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$

The Fourier Transform is then separable as well.

$$G(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx \int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy$$
$$= G_X(k_x)G_Y(k_y)$$

Example

$g(x,y) = \Pi(x)\Pi(y)$

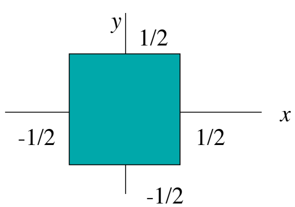
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

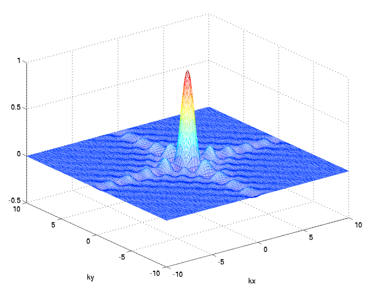
Example (sinc/rect)

Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$





Examples

Is this function separable?

$$g(x,y) = \exp(-j2\pi(8x+9y))\sin(28\pi x)$$

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Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

Linearity

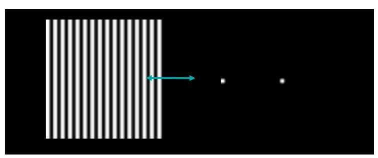
The Fourier Transform is linear.

$$F\left\{ag(x) + bh(x)\right\} = aG(k_x) + bH(k_x)$$

$$F\left[ag(x,y) + bh(x,y)\right] = aG(k_x,k_y) + bH(k_x,k_y)$$

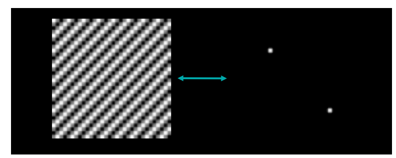
Examples

$$g(x,y) = 1 + e^{-j2\pi ax}$$
$$G(k_x,k_y) = \delta\left(k_x,k_y\right) + \delta(k_x + a)\delta(k_y)$$



$$g(x,y) = 1 + e^{j2\pi ay}$$
$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x)\delta(k_y - a)$$

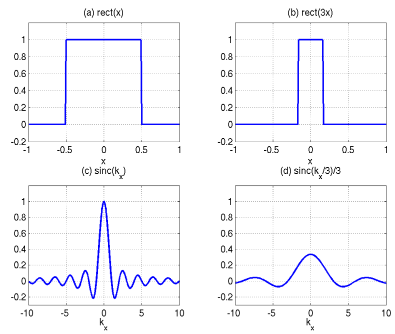
Examples



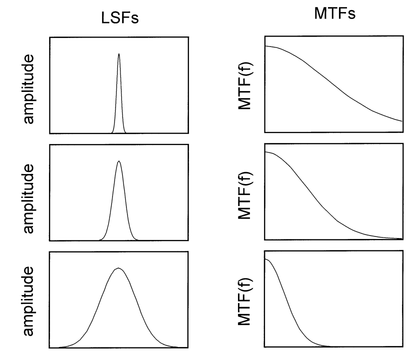
$$g(x,y) = \cos(2\pi(ax - by))$$
$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y + b) + \frac{1}{2}\delta(k_x + a)\delta(k_y - b)$$

Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right) \qquad F[g(ax,by)] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

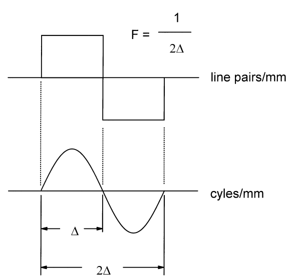


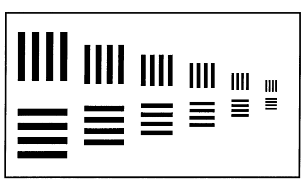
MTF = Fourier Transform of PSF

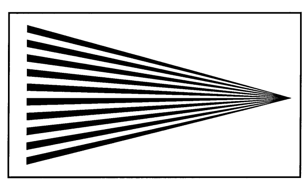


Bushberg et al 2001

Bushberg et al 2001



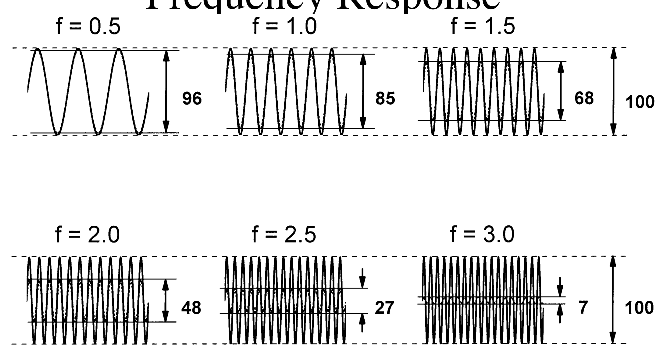




 Line Pair Test Phantom

Section of a Star Pattern

Modulation Transfer Function (MTF) or Frequency Response



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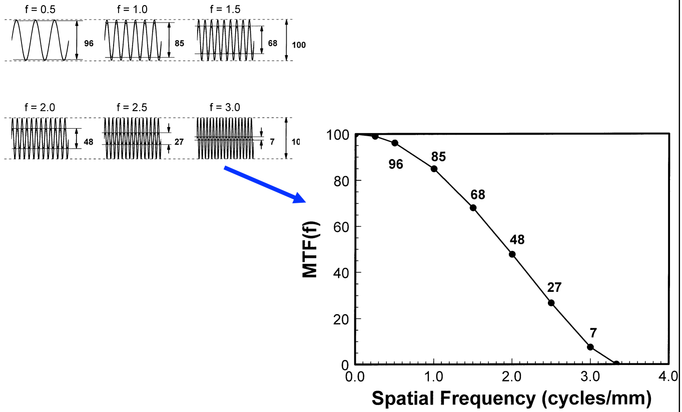
Modulation Transfer Function



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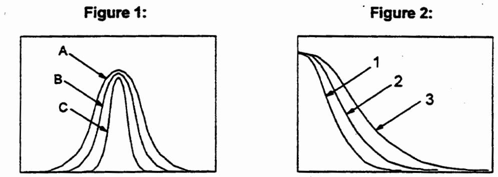

8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSF), which LSF will yield the best spatial resolution?

10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

A. MTF number 1
B. MTF number 2
C. MTF number 3

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Duality

Note the similarity between these two transforms

$$F\left\{e^{j2\pi a x}\right\} = \delta(k_x - a)$$
$$F\left\{\delta(x-a)\right\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\left\{G(x)\right\} = g(-k_x)$$

Application of Duality

$$F\left\{\mathrm{sinc}(x)\right\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\left\{\Pi(x)\right\} = \mathrm{sinc}(k_x)$.

Therefore from duality, $F\left\{\mathrm{sinc}(x)\right\} = \Pi(-k_x) = \Pi(k_x)$

Shift Theorem

$$F\{g(x-a)\} = G(k_x)e^{-j2\pi a k_x}$$

$$F[g(x-a, y-b)] = G(k_x, k_y)e^{-j2\pi(k_x a + k_y b)}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.

Each frequency component is shifted by a. This corresponds to a relative phase shift of

$-2\pi a/(\text{spatial period}) = -2\pi a k_x$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields $\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$

Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

$$e^{j2\pi k_x x} \longrightarrow \boxed{g(x)} \longrightarrow z(x)$$

$$\begin{aligned} z(x) &= g(x) * e^{j2\pi k_x x} \\ &= \int_{-\infty}^{\infty} g(u)e^{j2\pi k_x (x-u)}du \\ &= G(k_x)e^{j2\pi k_x x} \end{aligned}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

Convolution/Multiplication

Now consider an arbitrary input *h(x)*.

$$h(x) \longrightarrow \boxed{g(x)} \longrightarrow z(x)$$

Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x)e^{j2\pi k_x x}dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x)H(k_x)e^{j2\pi k_x x}dk_x$$

Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u) * h(x-u)du\right]e^{-j2\pi k_x x}dx \\ &= \int_{-\infty}^{\infty} g(u)\int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu \\ &= \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du \\ &= G(k_x)H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation $F\{g(x)h(x)\} = G(k_x) * H(k_x)$

2D Convolution/Multiplication

Convolution

$$F\left[g(x,y) ** h(x,y)\right] = G(k_x,k_y)H(k_x,k_y)$$

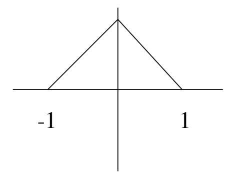
Multiplication

$$F\left[g(x,y)h(x,y)\right] = G(k_x,k_y) ** H(k_x,k_y)$$

Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

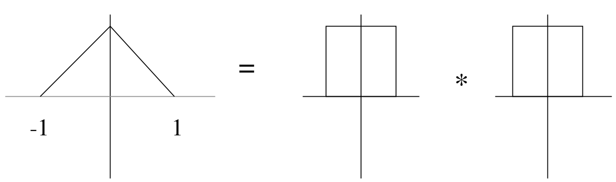
$$F(\Lambda(x)) = \int_{-1}^{1}(1-|x|)e^{-j2\pi k_x x}dx = ??$$



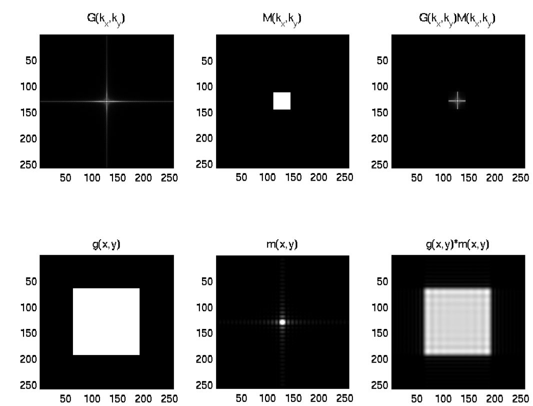
Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$

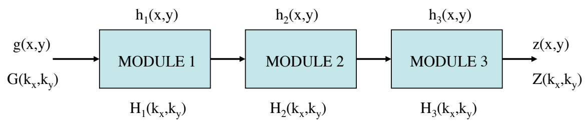
$$F(\Lambda(x)) = \sin c^2(k_x)$$



Convolution Example



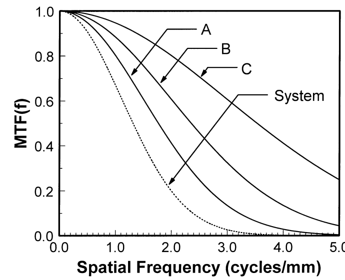
Response of an Imaging System



$z(x,y)=g(x,y)**h_1(x,y)**h_2(x,y)**h_3(x,y)$

$Z(k_x,k_y)=G(k_x,k_y)\ H_1(k_x,k_y)\ H_2(k_x,k_y)\ H_3(k_x,k_y)$

System MTF = Product of MTFs of Components



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Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots FWHM_N^2}$$

Example

$$FWHM_1 = 1mm$$

$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$

D74. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ______ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

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Modulation

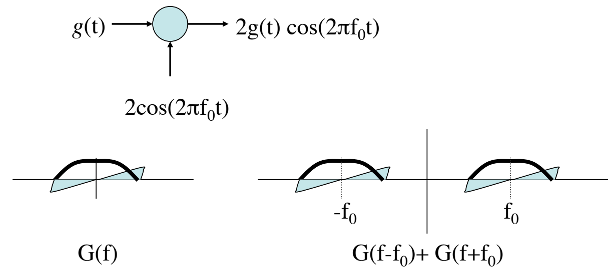
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

Example

Amplitude Modulation (e.g. AM Radio)



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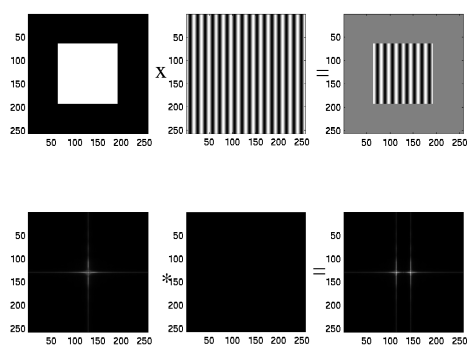
Modulation Example



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Summary of Basic Properties

Linearity

$$F\left[ag(x,y) + bh(x,y)\right] = aG(k_x,k_y) + bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

Duality

$$F\left\{G(x)\right\} = g(-k_x)$$

Shift

$$F\left[g(x-a,y-b)\right] = G(k_x,k_y)e^{-j2\pi(k_x a + k_y b)}$$

Convolution

$$F\left[g(x,y)**h(x,y)\right] = G(k_x,k_y)H(k_x,k_y)$$

Multiplication

$$F\left[g(x,y)h(x,y)\right] = G(k_x,k_y)**H(k_x,k_y)$$

Modulation

$$F\left[g(x,y)e^{j2\pi(xa+yb)}\right] = G(k_x - a, k_y - b)$$

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