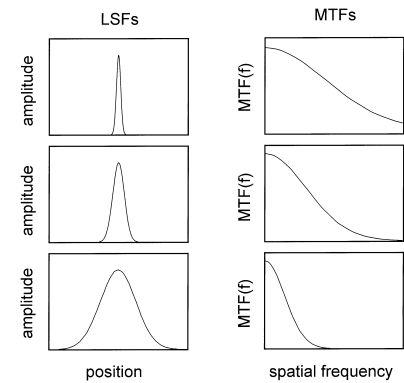


Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
CT/Fourier Lecture 4

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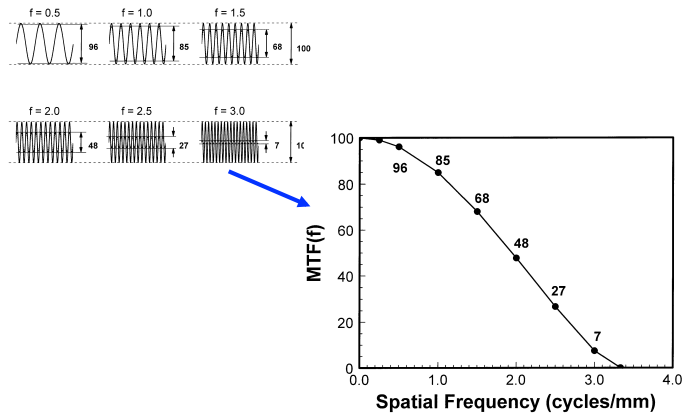
MTF = Fourier Transform of PSF



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Bushberg et al 2001

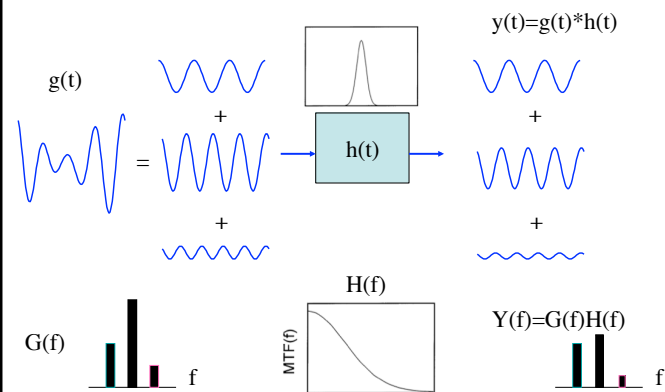
Modulation Transfer Function



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Bushberg et al 2001

Convolution Theorem



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Convolution/Modulation Theorem

$$\begin{aligned}
 F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx \\
 &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\
 &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\
 &= G(k_x) H(k_x)
 \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

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2D Convolution/Multiplication

Convolution

$$F[g(x,y) * h(x,y)] = G(k_x, k_y) H(k_x, k_y)$$

Multiplication

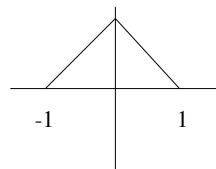
$$F[g(x,y)h(x,y)] = G(k_x, k_y) * H(k_x, k_y)$$

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Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1-|x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^1 (1-|x|) e^{-j2\pi k_x x} dx = ??$$

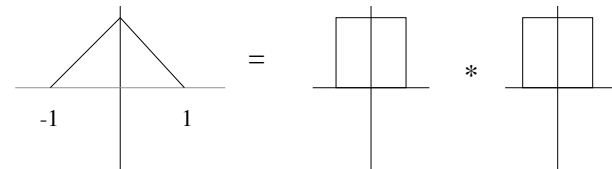


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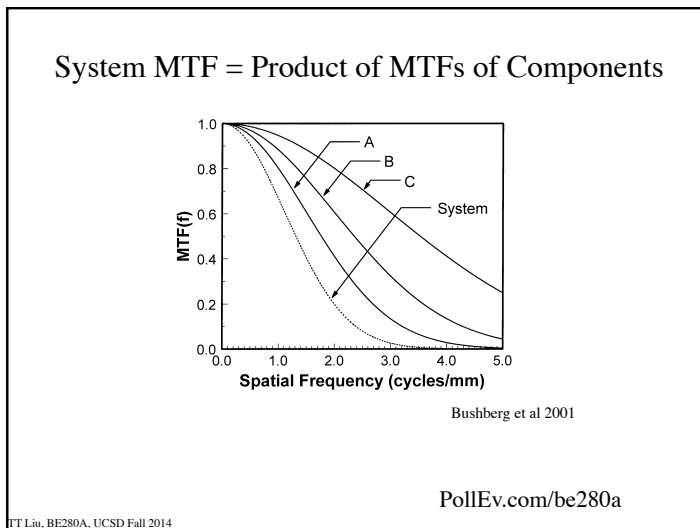
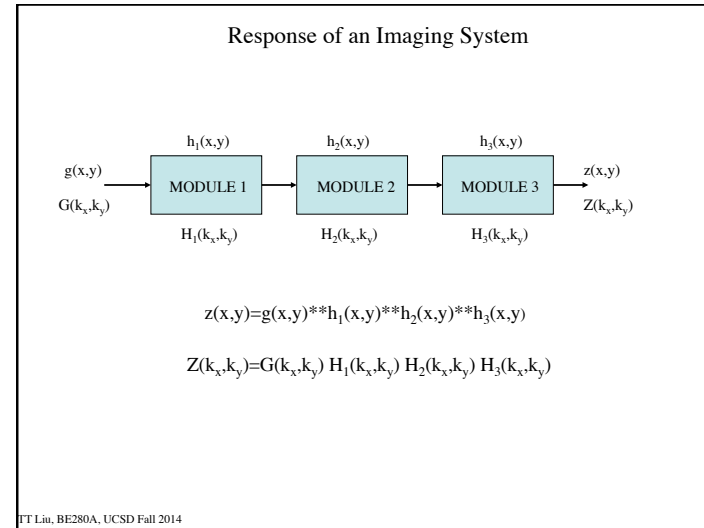
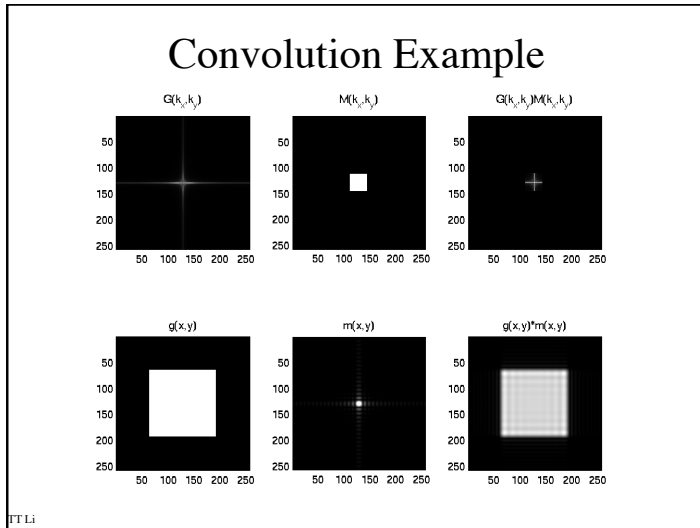
Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$

$$F(\Lambda(x)) = \text{sinc}^2(k_x)$$



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Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \dots + FWHM_N^2}$$

Example

$$FWHM_1 = 1mm$$

$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$

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- D74.** The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is _____ mm.
- A. 15
 - B. 11.2
 - C. 7.5
 - D. 5.0
 - E. 0.5

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Modulation

$$F[g(x)e^{j2\pi k_0 x}] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

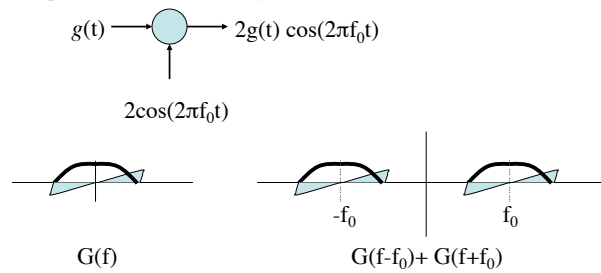
$$F[g(x)\cos(2\pi k_0 x)] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F[g(x)\sin(2\pi k_0 x)] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

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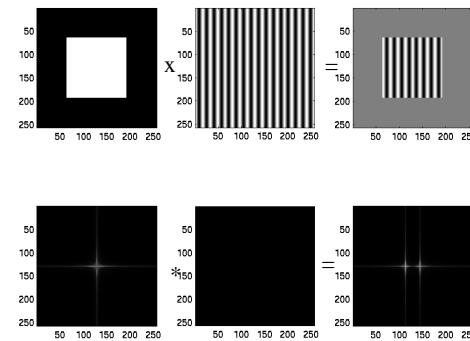
Example

Amplitude Modulation (e.g. AM Radio)



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Modulation Example



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Summary of Basic Properties

Linearity

$$F[ag(x,y) + bh(x,y)] = aG(k_x, k_y) + bH(k_x, k_y)$$

Scaling

$$F[g(ax, by)] = \frac{1}{|ab|} G\left(\frac{k_x}{a}, \frac{k_y}{b}\right)$$

Duality

$$F\{G(x)\} = g(-k_x)$$

Shift

$$F[g(x-a, y-b)] = G(k_x, k_y) e^{-j2\pi(k_x a + k_y b)}$$

Convolution

$$F[g(x,y) * h(x,y)] = G(k_x, k_y) H(k_x, k_y)$$

Multiplication

$$F[g(x,y)h(x,y)] = G(k_x, k_y) * H(k_x, k_y)$$

Modulation

$$F[g(x,y)e^{j2\pi(ux+vy)}] = G(k_x - a, k_y - b)$$

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In-class Exercise

$$\mu(x, y) = \text{rect}(x, y) \cos(2\pi(x + y))$$

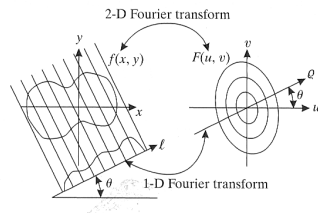
Sketch the object

Calculate and sketch the Fourier Transform of the Object

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Projection-Slice Theorem

$$\begin{aligned} G(\rho, \theta) &= \int_{-\infty}^{\infty} g(l, \theta) e^{-j2\pi\rho l} dl \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - l) e^{-j2\pi\rho l} dx dy dl \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\rho(x \cos \theta + y \sin \theta)} dx dy \\ &= F_{2D}[f(x, y)]|_{u=\rho \cos \theta, v=\rho \sin \theta} \end{aligned}$$



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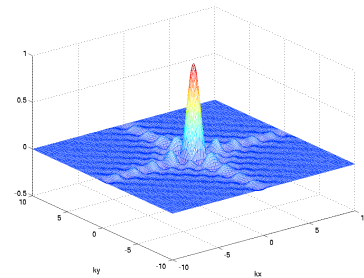
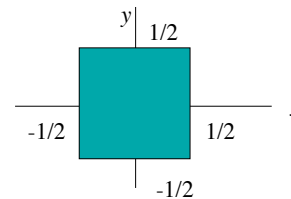
Prince&Links 2006

Example (sinc/rect)

Example

$$g(x, y) = \Pi(x)\Pi(y)$$

$$G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$$



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Projection-Slice Theorem

$$\begin{aligned}
 U(k_x, 0) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) e^{-j2\pi(k_x x + k_y y)} dx dy \\
 &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(x, y) dy \right] e^{-j2\pi k_x x} dx \\
 &= \int_{-\infty}^{\infty} g(x, 0) e^{-j2\pi k_x x} dx \\
 &= \int_{-\infty}^{\infty} g(l, 0) e^{-j2\pi k l} dl
 \end{aligned}$$

In-Class Example:
 $\mu(x, y) = \cos 2\pi x$

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Projection-Slice Theorem

$$\begin{aligned}
 U(k_x, k_y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) e^{-j2\pi(k_x x + k_y y)} dx dy \\
 &= F_{2D}[u(x, y)]
 \end{aligned}$$

$$\begin{aligned}
 U(k_x, k_y) &= G(k, \theta) \\
 k_x &= k \cos \theta \\
 k_y &= k \sin \theta \\
 k &= \sqrt{k_x^2 + k_y^2}
 \end{aligned}$$

$$G(k, \theta) = \int_{-\infty}^{\infty} g(l, \theta) e^{-j2\pi k l} dl$$

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Projection-Slice Theorem

$$\begin{aligned}
 G(\rho, \theta) &= \int_{-\infty}^{\infty} g(l, \theta) e^{-j2\pi \rho l} dl \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - l) e^{-j2\pi \rho l} dx dy dl \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \rho (x \cos \theta + y \sin \theta)} dx dy \\
 &= F_{2D}[f(x, y)] \Big|_{u = \rho \cos \theta, v = \rho \sin \theta}
 \end{aligned}$$

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In-class Exercise

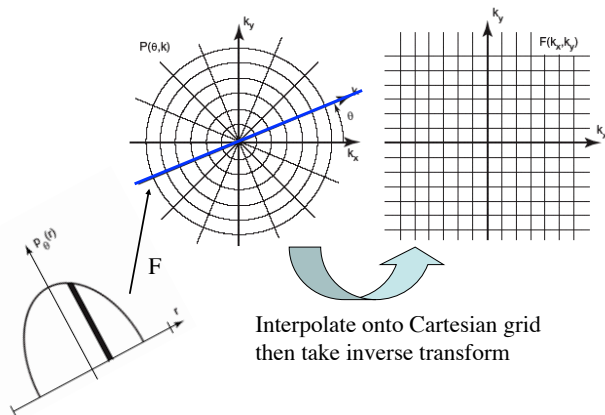
$$\mu(x, y) = \text{rect}(x, y) \cos(2\pi(x + y))$$

Sketch this object.
 What are the projections at theta = 0 and 90 degrees?
 For what angle is the projection maximized?

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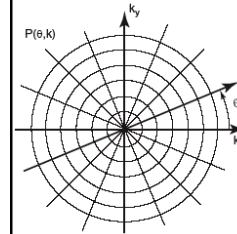
Fourier Reconstruction



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Suetens 2002

Polar Version of Inverse FT



$$\begin{aligned} \mu(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y \\ &= \int_0^{2\pi} \int_0^{\infty} G(k, \theta) e^{j2\pi(xk \cos\theta + yk \sin\theta)} k dk d\theta \\ &= \int_0^{\pi} \int_{-\infty}^{\infty} G(k, \theta) e^{j2\pi k(x \cos\theta + y \sin\theta)} |k| dk d\theta \end{aligned}$$

Note :

$$g(l, \theta + \pi) = g(-l, \theta)$$

So

$$G(k, \theta + \pi) = G(-k, \theta)$$

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Filtered Backprojection

$$\begin{aligned} \mu(x, y) &= \int_0^{\pi} \int_{-\infty}^{\infty} G(k, \theta) e^{j2\pi(xk \cos\theta + yk \sin\theta)} |k| dk d\theta \\ &= \int_0^{\pi} \int_{-\infty}^{\infty} |k| G(k, \theta) e^{j2\pi k l} dk d\theta \\ &= \int_0^{\pi} g^*(l, \theta) d\theta \quad \leftarrow \text{Backproject a filtered projection} \end{aligned}$$

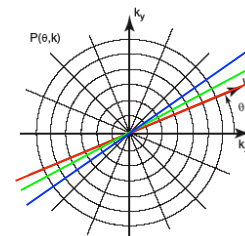
where $l = x \cos\theta + y \sin\theta$

$$\begin{aligned} g^*(l, \theta) &= \int_{-\infty}^{\infty} |k| G(k, \theta) e^{j2\pi k l} dk \\ &= g(l, \theta) * F^{-1}[|k|] \\ &= g(l, \theta) * q(l) \end{aligned}$$

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Fourier Interpretation



$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

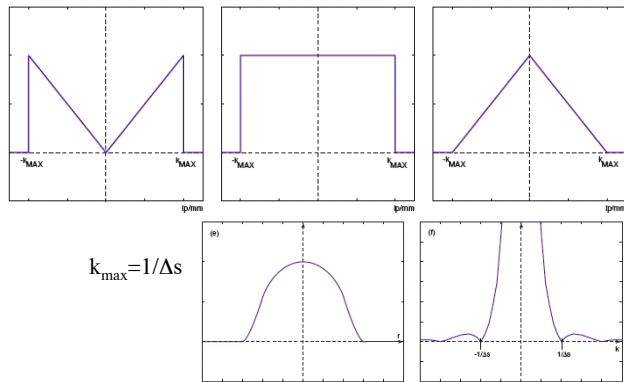
Low frequencies are oversampled. So to compensate for this, multiply the k-space data by $|k|$ before inverse transforming.



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Kak and Slaney; Suetens 2002

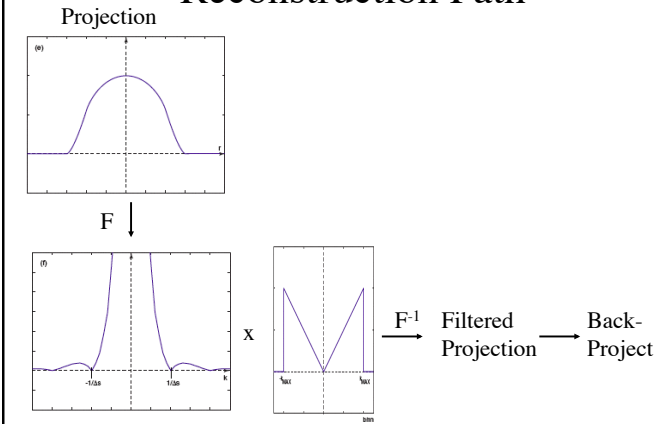
Ram-Lak Filter



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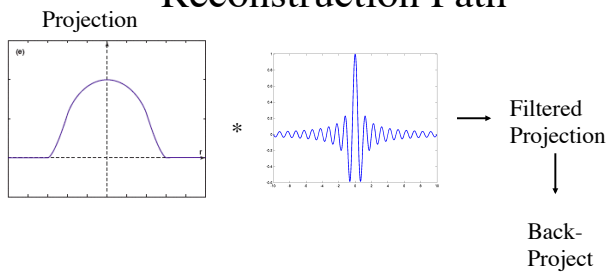
Reconstruction Path



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Suetens 2002

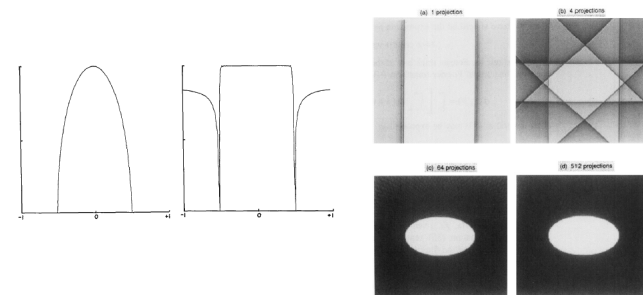
Reconstruction Path



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Suetens 2002

Example



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Kak and Slaney

Example

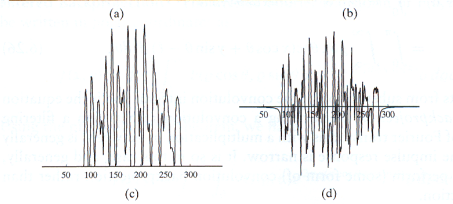
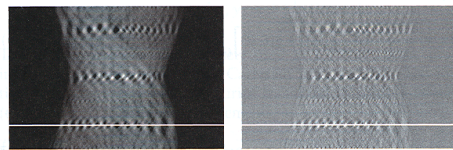


Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row
[white line in (a)]; and
(d) profile of filtered
sinogram row [white line in
(b)].

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Prince and Links 2005

Example

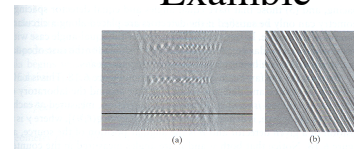


Figure 6.16
Backprojection step.

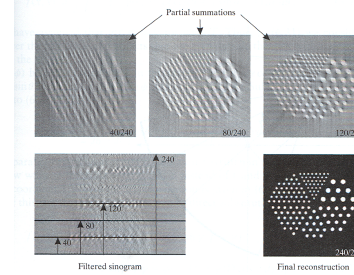


Figure 6.17
Summation step.

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Prince and Links 2005