Projection-Slice Theorem

\[ G(\rho, \theta) = \int_{-\infty}^{\infty} g(l, \theta) e^{-j2\pi \rho l} \, dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - l) e^{-j2\pi \rho (x \cos \theta + y \sin \theta)} \, dx \, dy \]

\[ = F_{2D}[f(x, y)] |_{x \cos \theta = \rho \cos \theta} |_{y \sin \theta = \rho \sin \theta} \]

Example (sinc/rect)

Example

\[ g(x, y) = \Pi(x) \Pi(y) \]

\[ G(k_x, k_y) = \text{sinc}(k_x) \text{sinc}(k_y) \]

Projection at \( \theta = 0 \):

\[ g(l, 0) = \text{rect}(l) \rightarrow F(g(l, 0)) = \text{sinc}(k) \]

\[ k_x = k \cos \theta = k; \quad k_y = k \sin \theta = 0 \]

\[ G(k_x, k_y) = \text{sinc}(k) \text{sinc}(0) = \text{sinc}(k) \]

Projection at \( \theta = 90 \):

\[ g(l, 90) = \text{rect}(l) \rightarrow F(g(l, 90)) = \text{sinc}(k) \]

\[ k_x = k \cos \theta = 0; \quad k_y = k \sin \theta = k \]

\[ G(k_x, k_y) = \text{sinc}(0) \text{sinc}(k) = \text{sinc}(k) \]

Example (sinc/rect)
Example (sinc/rect)

\[ g(x, y) = \Pi(x)\Pi(y) \]
\[ G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y) \]

Projection at \( \theta = 45 \),
\[ g(l, 45) = \sqrt{2}\Lambda \left( \frac{l}{\sqrt{2}/2} \right) = 2\text{rect}\left( \frac{l}{\sqrt{2}/2} \right) \]
\[ \rightarrow F(g(l, 45)) = 2\sqrt{2} \text{sinc} \left( k_x \frac{\sqrt{2}}{2} \right) \text{sinc} \left( k_y \frac{\sqrt{2}}{2} \right) = \text{sinc} \left( k \frac{\sqrt{2}}{2} \right) \]

where \( k_x = k \cos\theta = k \frac{\sqrt{2}}{2}; \ k_y = k \sin\theta = k \frac{\sqrt{2}}{2} \)
\[ G(k_x, k_y) = \text{sinc} \left( k_x \frac{\sqrt{2}}{2} \right) \text{sinc} \left( k_y \frac{\sqrt{2}}{2} \right) = \text{sinc} \left( k \frac{\sqrt{2}}{2} \right) \]
In-Class Example:

\[
\mu(x, y) = \cos 2\pi x
\]

In-class Exercise

\[
\mu(x, y) = \text{rect}(x, y)\cos(2\pi(x + y))
\]

Sketch this object.

What are the projections at theta = 0 and 90 degrees?

For what angle is the projection maximized?

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**Fourier Reconstruction**

Interpolate onto Cartesian grid then take inverse transform

**Polar Version of Inverse FT**

\[
\mu(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{i(2\pi k_x x + 2\pi k_y y)} \, dk_x \, dk_y
\]

\[
= \int_{0}^{2\pi} \int_{0}^{\infty} G(k, \theta) e^{i(2\pi k \cos \theta \, x + 2\pi k \sin \theta \, y)} \, k \, dk \, d\theta
\]

\[
= \int_{0}^{\infty} \int_{0}^{\pi} G(k, \theta) e^{i(2\pi k \cos \theta \, x + 2\pi k \sin \theta \, y)} \, |k| \, dk \, d\theta
\]

Note:

\[g(l, \theta + \pi) = g(-l, \theta)\]

So

\[G(k, \theta + \pi) = G(-k, \theta)\]

**Filtered Backprojection**

\[
\mu(x, y) = \int_{0}^{\infty} \int_{0}^{\pi} G(k, \theta) e^{i(2\pi k \cos \theta \, x + 2\pi k \sin \theta \, y)} \, |k| \, dk \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{0}^{\infty} |k| G(k, \theta) e^{i 2\pi kl} \, dk \, d\theta
\]

where \( l = x \cos \theta + y \sin \theta \)

\[g^*(l, \theta) = \int_{0}^{\infty} |k| G(k, \theta) e^{i 2\pi kl} \, dk\]

\[= g(l, \theta) * F^{-1}[|k|] = g(l, \theta) * q(l)\]

**Fourier Interpretation**

\[\text{Density} = \frac{N}{\text{circumference}} = \frac{N}{2\pi|k|}\]

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by \(|k|\) before inverse transforming.
Ram-Lak Filter

\[ k_{\text{max}} = \frac{1}{\Delta s} \]

Reconstruction Path

Projection

\[ F \]

Filtered Projection

Back-Project

Example

Kak and Slaney
Example

Figure 6.15
Convolution steps:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Example

Figure 6.56
Backprojection step.

Figure 6.57
Substitution step.