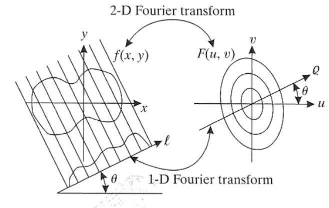
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
CT/Fourier Lecture 5

Projection-Slice Theorem

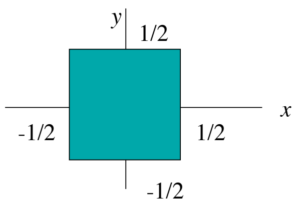
$$
\begin{aligned}
G(\rho,\theta) &= \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy \\
&= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}
\end{aligned}
$$

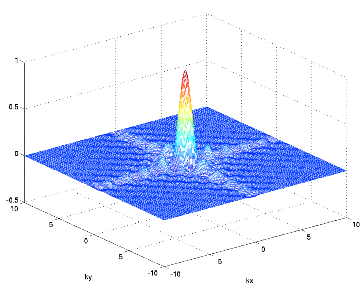


Prince&Links 2006

Example (sinc/rect)

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$





Example (sinc/rect)

Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Projection at $\theta = 0$;

$g(l,0) = rect(l) \rightarrow F(g(l,0)) = \mathrm{sinc(k)}$

$k_x = k\cos\theta = k;\ \ k_y = k\sin\theta = 0$

$G(k_x,k_y) = \mathrm{sinc}(k)\mathrm{sinc}(0) = \mathrm{sinc}(k)$

Projection at $\theta = 90$;

$g(l,90) = rect(l) \rightarrow F(g(l,90)) = \mathrm{sinc(k)}$

$k_x = k\cos\theta = 0;\ \ k_y = k\sin\theta = k$

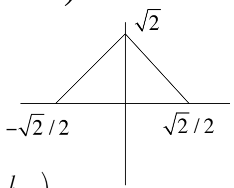
$G(k_x,k_y) = \mathrm{sinc}(0)\mathrm{sinc}(k) = \mathrm{sinc}(k)$

Example (sinc/rect)

Example

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$



Projection at $\theta = 45$;

$$g(l,45) = \sqrt{2}\Lambda\left(\frac{l}{\sqrt{2}/2}\right) = 2rect\left(\frac{l}{\sqrt{2}/2}\right) * rect\left(\frac{l}{\sqrt{2}/2}\right)$$

$$\rightarrow F(g(l,45)) = 2\frac{\sqrt{2}}{2}\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right) = \mathrm{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$

$$k_x = k\cos\theta = k\frac{\sqrt{2}}{2};\quad k_y = k\sin\theta = k\frac{\sqrt{2}}{2}$$

$$G(k_x,k_y) = \mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right) = \mathrm{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$

Example (sinc/rect)

Example

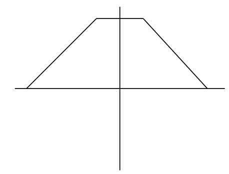
$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Projection at $0 < \theta < 90$

$k_x = k\cos\theta;\quad k_y = k\sin\theta$

$G(k_x,k_y) = \mathrm{sinc}(k\cos\theta)\mathrm{sinc}(k\sin\theta)$

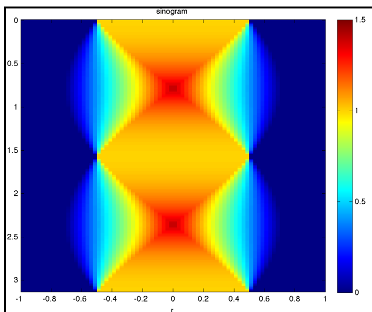


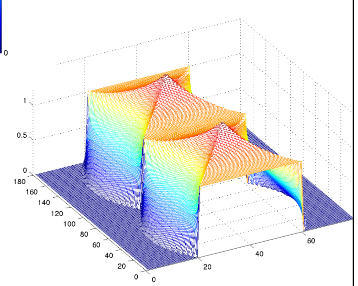
$$g(l,\theta) = F^{-1}\left(\mathrm{sinc}(k\cos\theta)\mathrm{sinc}(k\sin\theta)\right)$$

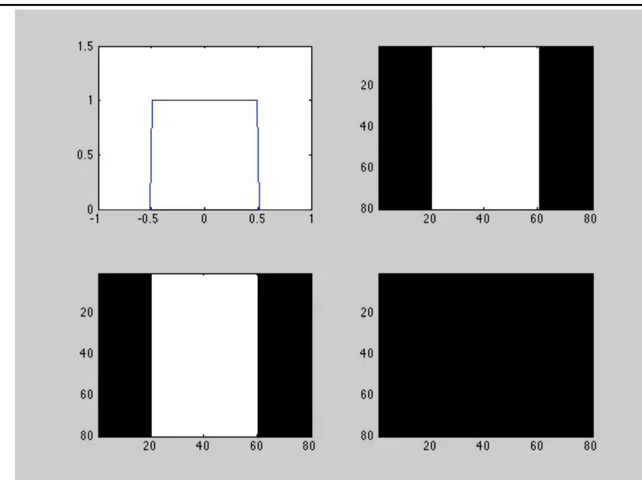
$$= F^{-1}\left(\mathrm{sinc}(k\cos\theta)\right) * F^{-1}\left(\mathrm{sinc}(k\sin\theta)\right)$$

$$= \frac{1}{\cos\theta}rect\left(\frac{l}{\cos\theta}\right) * \frac{1}{\sin\theta}rect\left(\frac{l}{\sin\theta}\right)$$

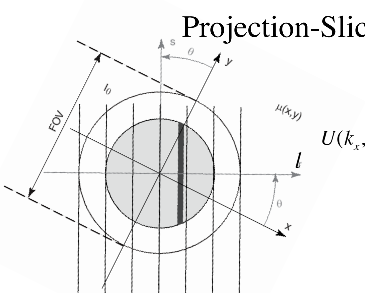
$$= \frac{1}{\cos\theta\sin\theta}rect\left(\frac{l}{\cos\theta}\right) * rect\left(\frac{l}{\sin\theta}\right)$$



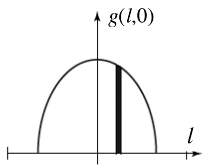




Projection-Slice Theorem



$$U(k_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{-\infty}\left[\int_{-\infty}^{-\infty}\mu(x,y)dy\right]e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{-\infty}g(x,0)e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{-\infty}g(l,0)e^{-j2\pi kl}dl$$

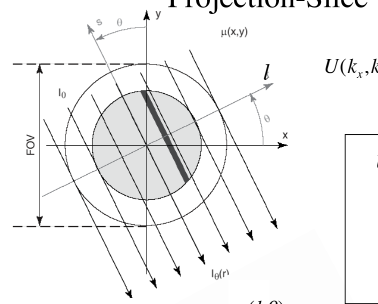


In-Class Example:

$\mu(x,y) = \cos 2\pi x$

Suetens 2002

Projection-Slice Theorem



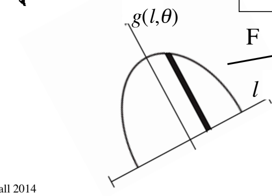
$$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= F_{2D}\left[\mu(x,y)\right]$$

$U(k_x,k_y) = G(k,\theta)$

$k_x = k\cos\theta$

$k_y = k\sin\theta$

$k = \sqrt{k_x^2 + k_y^2}$

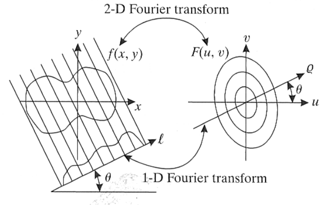


F → $G(k,\theta) = \int_{-\infty}^{\infty}g(l,\theta)e^{-j2\pi kl}dl$

Suetens 2002

Projection-Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty}g(l,\theta)e^{-j2\pi\rho l}dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\delta(x\cos\theta + y\sin\theta - l)e^{-j2\pi\rho l}dx\,dy\,dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi\rho(x\cos\theta + y\sin\theta)}dx\,dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$



Prince&Links 2006

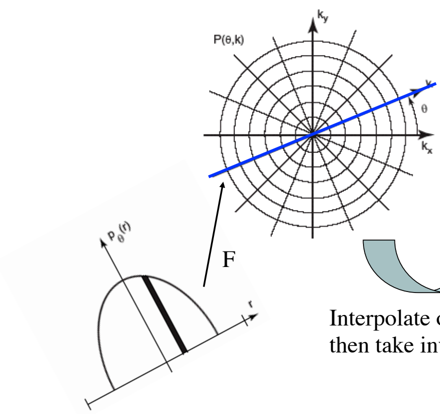
In-class Exercise

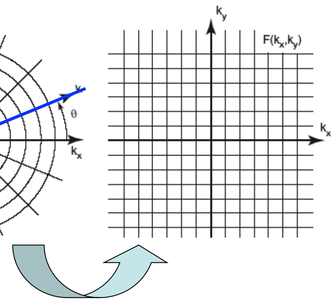
$$\mu(x,y) = rect(x,y)\cos\left(2\pi(x+y)\right)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

PollEv.com/be280a

Fourier Reconstruction

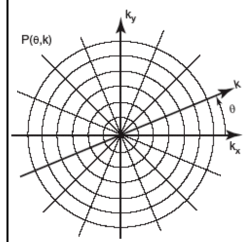




Interpolate onto Cartesian grid
then take inverse transform

 Suetens 2002

Polar Version of Inverse FT



$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta) e^{j2\pi(xk\cos\theta + yk\sin\theta)} k\,dk\,d\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta) e^{j2\pi k(x\cos\theta + y\sin\theta)} |k|\,dk\,d\theta$$

Note:

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

 Suetens 2002

Filtered Backprojection

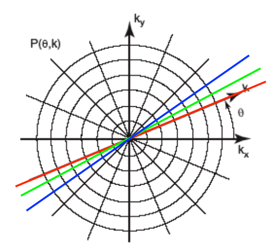
$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta) e^{j2\pi(xk\cos\theta + yk\sin\theta)} |k|\,dk\,d\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k| G(k,\theta) e^{j2\pi kl} dk\,d\theta$$
$$= \int_0^{\pi} g^*(l,\theta)\,d\theta \quad \longleftarrow$$ Backproject a filtered projection

$where\ l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty} |k| G(k,\theta) e^{j2\pi kl} dk$$
$$= g(l,\theta) * F^{-1}\big[|k|\big]$$
$$= g(l,\theta) * q(l)$$

 Suetens 2002

Fourier Interpretation

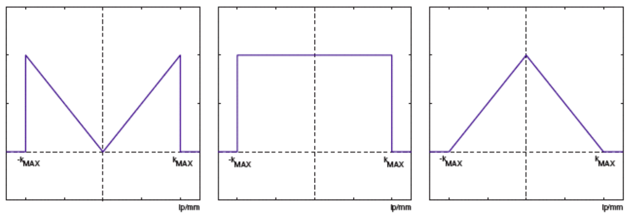


$$\text{Density} \approx \frac{\text{N}}{\text{circumference}} \approx \frac{\text{N}}{2\pi|\text{k}|}$$

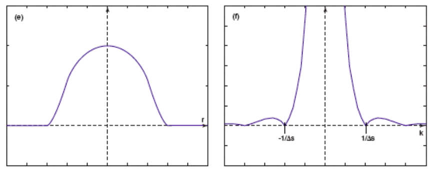
Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

Ram-Lak Filter



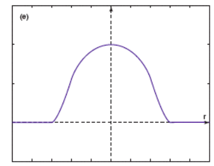
$k_{max}=1/\Delta s$



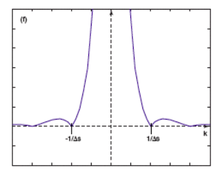
Suetens 2002

Reconstruction Path

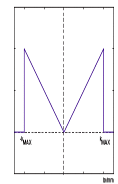
Projection



F



x

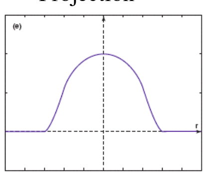


F^{-1} Filtered Projection → Back-Project

Suetens 2002

Reconstruction Path

Projection



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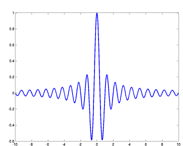


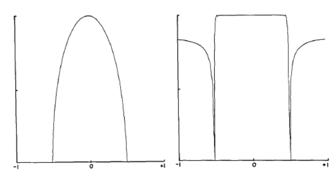
→ Filtered Projection

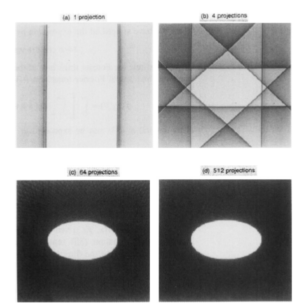
↓

Back-Project

Suetens 2002

Example





Kak and Slaney

Example

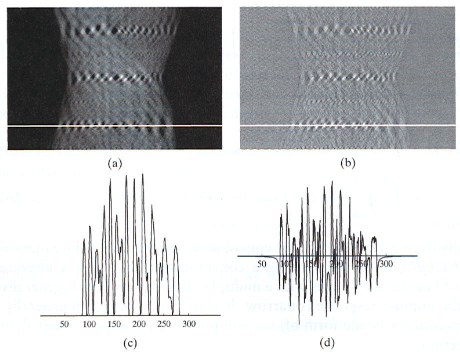


Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Prince and Links 2005

Example

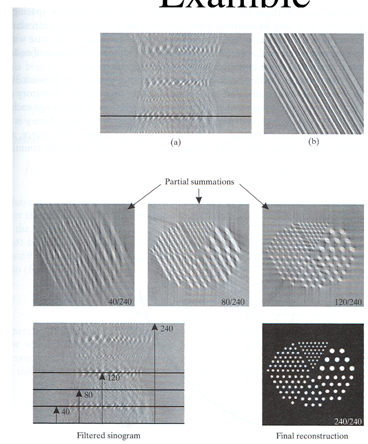


Figure 6.16
Backprojection step.

Figure 6.17
Summation step.

Prince and Links 2005