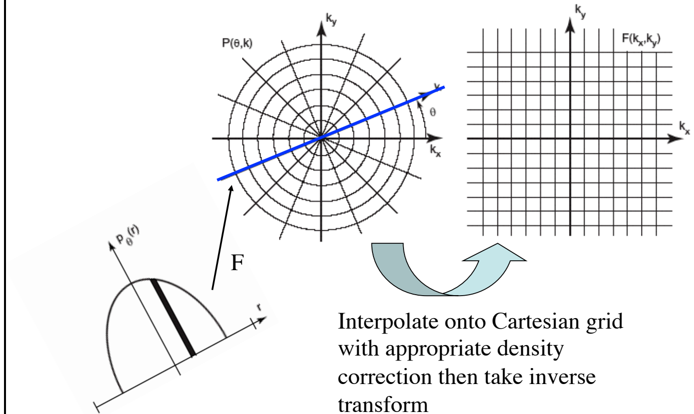
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
CT/Fourier Lecture 5

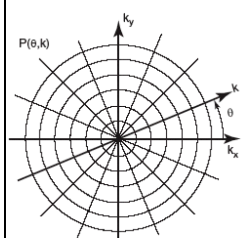
Fourier Reconstruction



Interpolate onto Cartesian grid with appropriate density correction then take inverse transform

Suetens 2002

Polar Version of Inverse FT



$$
\begin{aligned}
\mu(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x+k_y y)}dk_x dk_y \\
&= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta+yk\sin\theta)}kdkd\theta \\
&= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta+y\sin\theta)}|k|dkd\theta
\end{aligned}
$$

Note:

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

Suetens 2002

Filtered Backprojection

$$
\begin{aligned}
\mu(x,y) &= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta\ +yk\sin\theta)}|k|dkd\theta \\
&= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dkd\theta \\
&= \int_0^{\pi} g^*(l,\theta)d\theta
\end{aligned}
$$

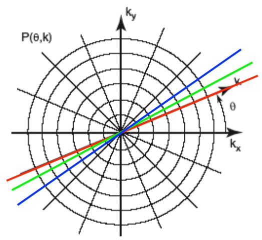
← Backproject a filtered projection

where $l = x\cos\theta + y\sin\theta$

$$
\begin{aligned}
g^*(l,\theta) &= \int_{-\infty}^{\infty}|k|G(k,\theta)e^{j2\pi kl}dk \\
&= g(l,\theta) * F^{-1}\left[|k|\right] \\
&= g(l,\theta) * q(l)
\end{aligned}
$$

Suetens 2002

Fourier Interpretation

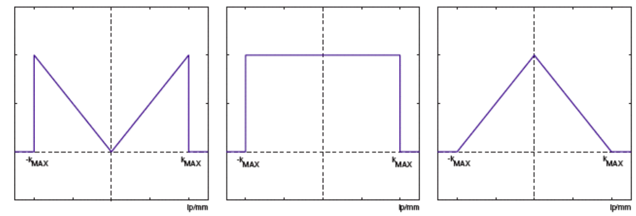


$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

Ram-Lak Filter

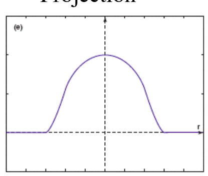


Typically will pick $k_{max}=1/\Delta s$ where Δs is the beamwidth

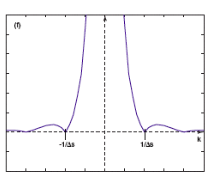
Suetens 2002

Reconstruction Path

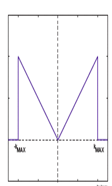
Projection



F



x

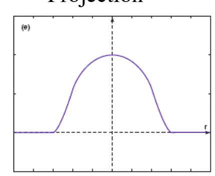


F^{-1} → Filtered Projection → Back-Project

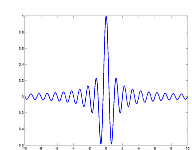
Suetens 2002

Reconstruction Path

Projection



*

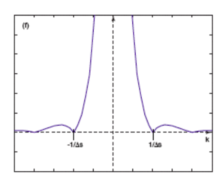


→ Filtered Projection

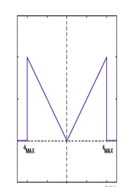
↓

Back-Project

F



x

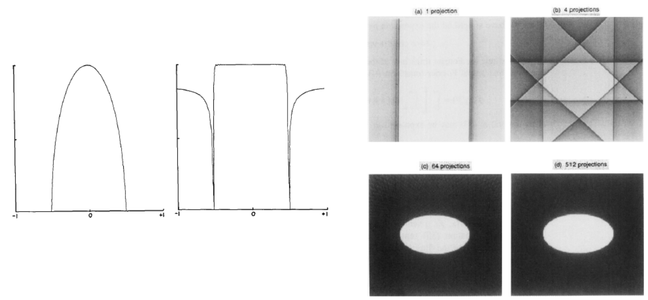


F^{-1} (↑)

F^{-1} → Filtered Projection ↑

Suetens 2002

Example



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Kak and Slaney

Example

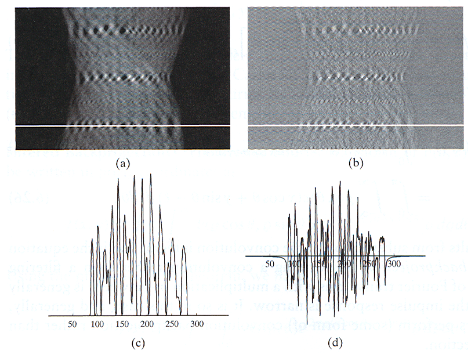

Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Prince and Links 2005

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Example

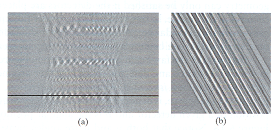

Figure 6.16
Backprojection step.

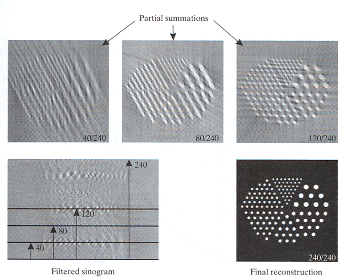

Figure 6.17
Summation step.

Prince and Links 2005

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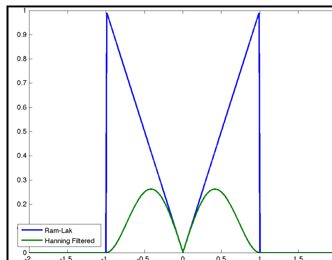
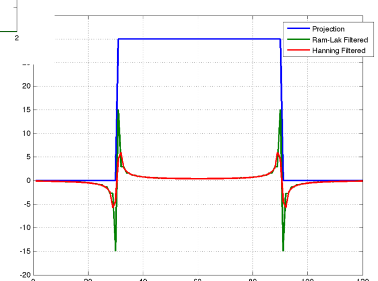


Filters

Filtered Projections



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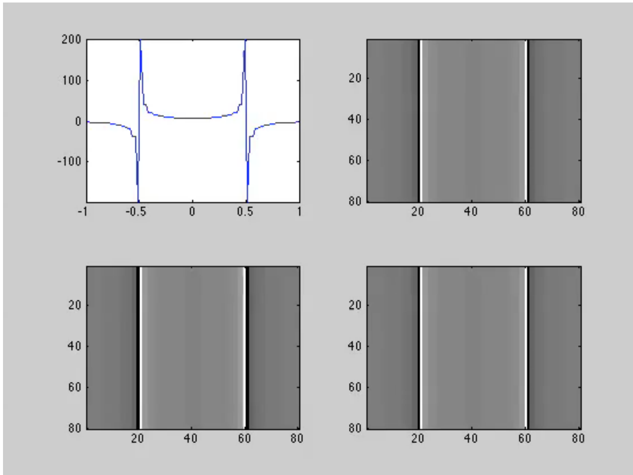
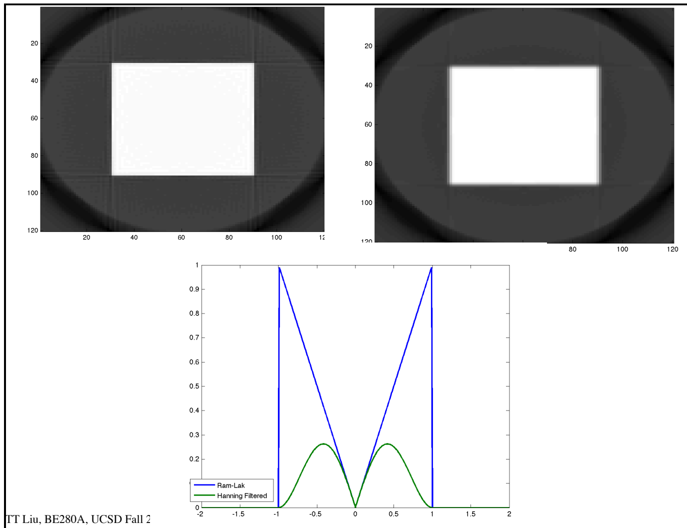


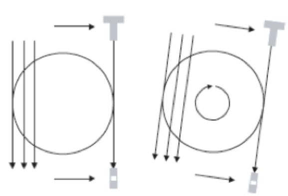


CT Sampling Requirements

What should the size of the detectors be?

How many detectors (or lines) do we need?

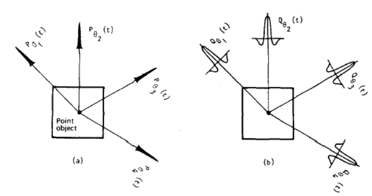
How many views do we need?

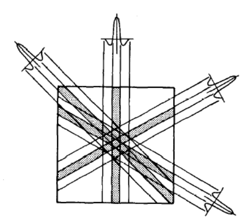


(a)

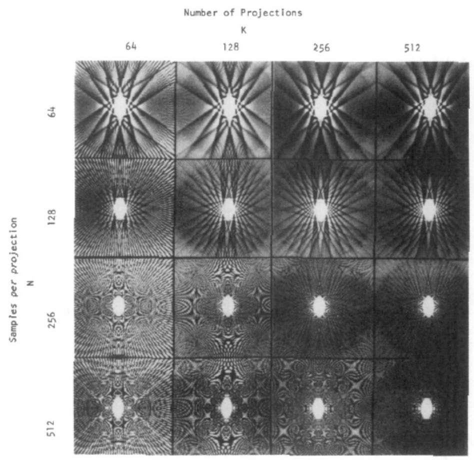
Suetens 2002

View Aliasing





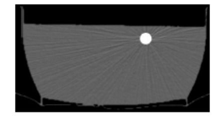
Kak and Slaney



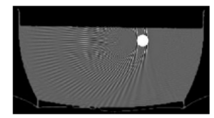
Kak and Slaney

Artifacts

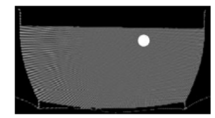
Object



Effect of Noise



Aliasing due to insufficient number of detectors



Aliasing due to insufficient number of views

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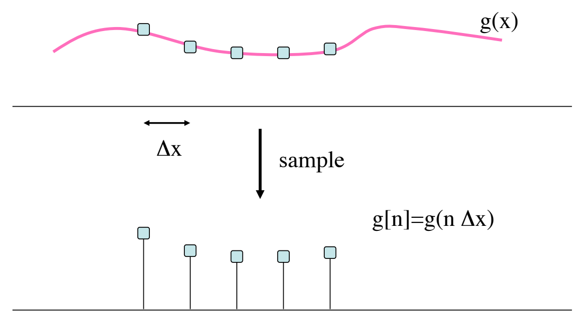
Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

The Process of Sampling



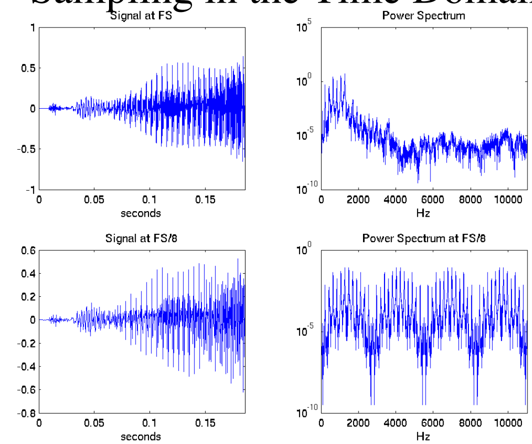
Questions

How finely do we need to sample?

What happens if we don't sample finely enough?

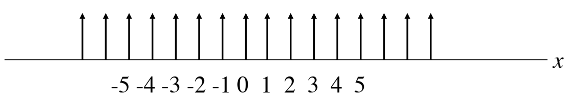
Can we reconstruct the original signal or image from its samples?

Sampling in the Time Domain



Comb Function

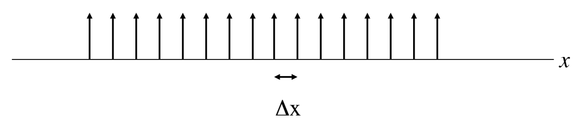
$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$
$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$
$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x)\sum_{n=-\infty}^{\infty}\delta(x - n\Delta x)$$

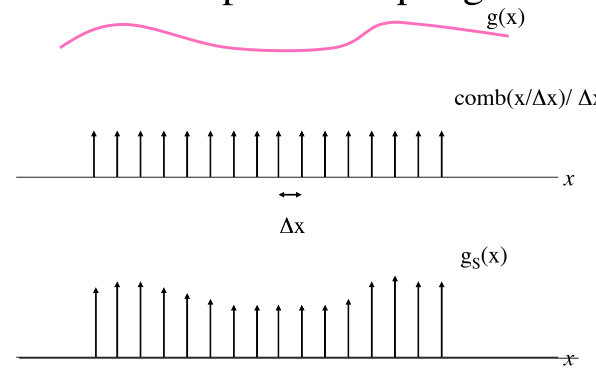
$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty}\delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

1D spatial sampling



Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$

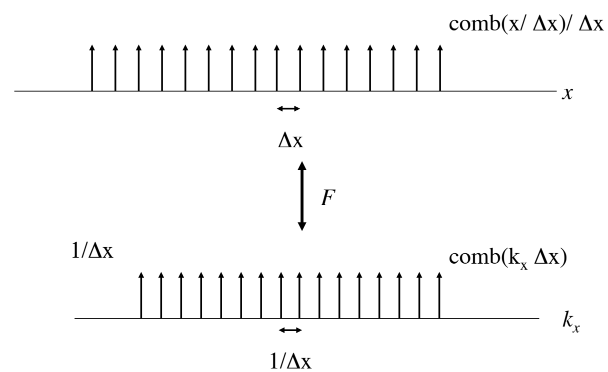
$$= \sum_{n=-\infty}^{\infty}\delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x}comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x}\Delta x comb(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty}\delta(k_x\Delta x - n)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta(k_x - \frac{n}{\Delta x})$$

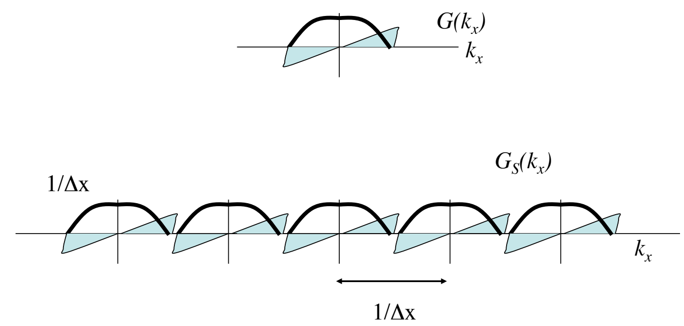
Fourier Transform of comb(x/ Δx)



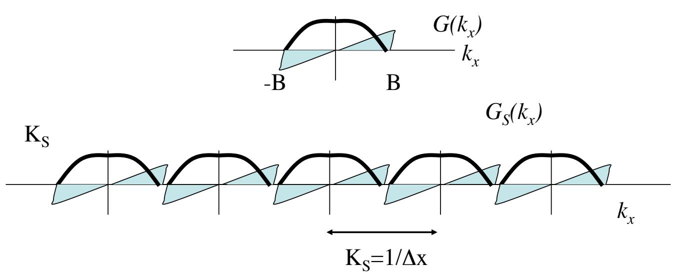
Fourier Transform of $g_S(x)$

$$
\begin{aligned}
F[g_S(x)] &= F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} G\left(k_x - \frac{n}{\Delta x}\right)
\end{aligned}
$$

Fourier Transform of $g_S(x)$



Nyquist Condition



To avoid overlap, we require that $1/\Delta x > 2B$ or
$K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

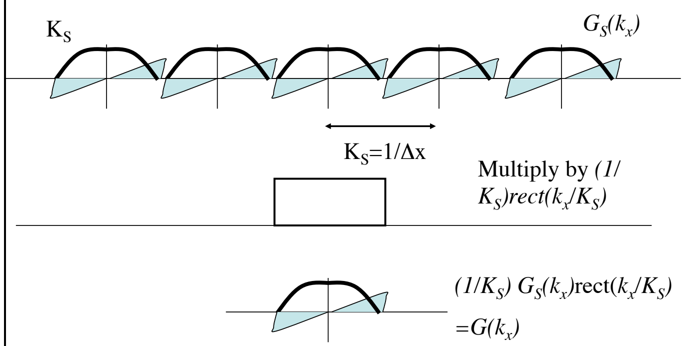
Example

Assume that the highest spatial frequency in an object is $B = 2$ cm^{-1}.
Thus, smallest spatial period is 0.5 cm.

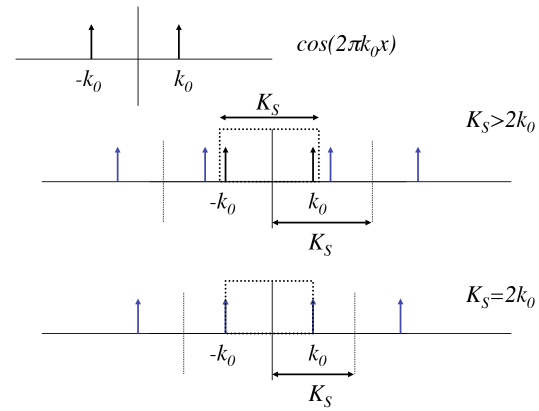
Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

Reconstruction from Samples



Example Cosine Reconstruction



Reconstruction from Samples

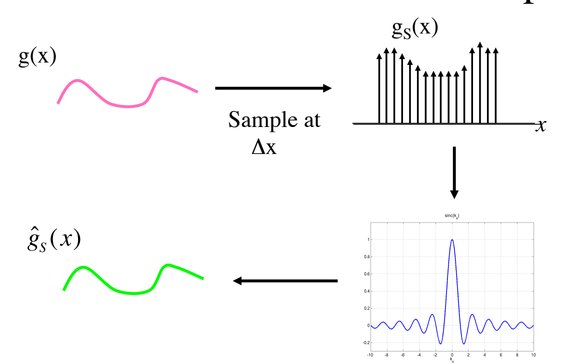
If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x/K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

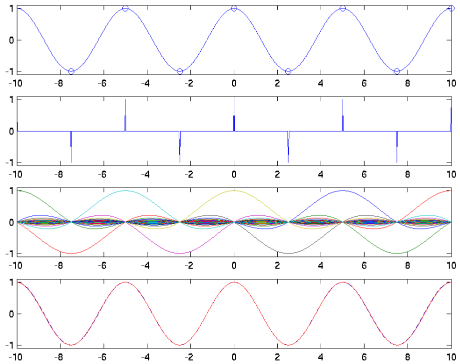
$$\begin{aligned}
\hat{g}_S(x) &= g_S(x) * \text{sin}c(K_s x) \\
&= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X) \right) * \text{sin}c(K_s x) \\
&= \sum_{n=-\infty}^{\infty} g(n\Delta X)\, \text{sin}c(K_s(x - n\Delta x))
\end{aligned}$$

Reconstruction from Samples

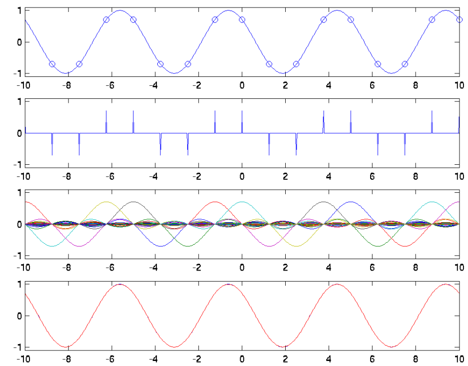


$$\text{sin}c(K_s x) = \text{sin}c(x/\Delta x)$$

Cosine Example with $K_S=2k_0$



Example with $K_s=4k_0$



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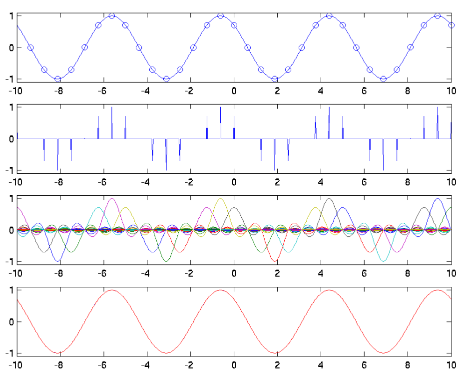
Example with $K_s=8k_0$



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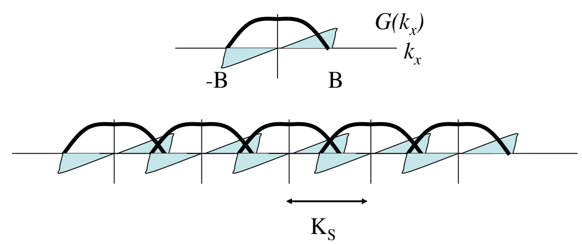
Aliasing



Aliasing occurs when the Nyquist condition is not satisfied.
This occurs for $K_S \leq 2B$

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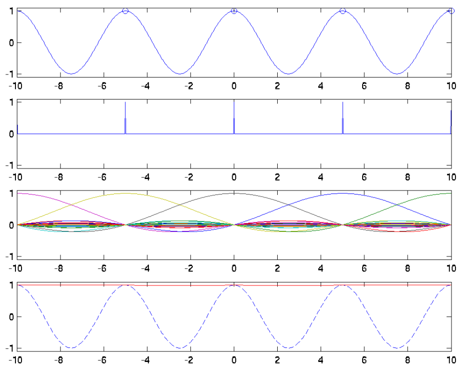
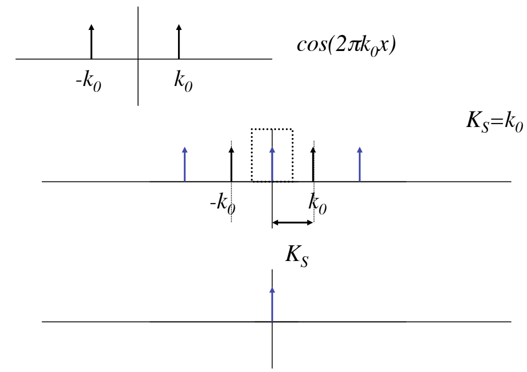
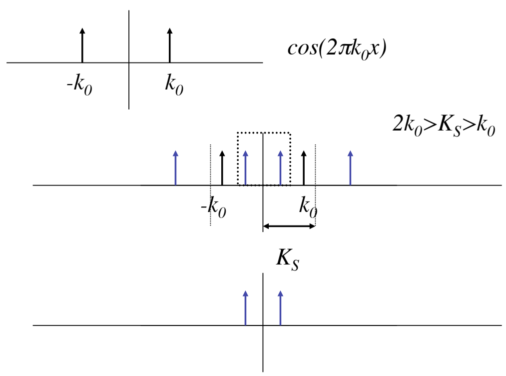
Aliasing Example

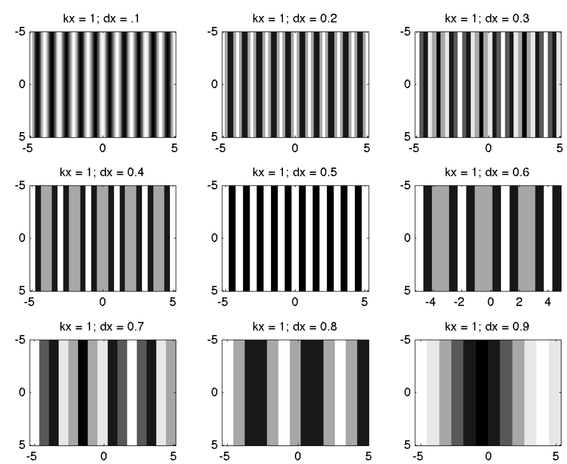


Aliasing Example



Aliasing Example





Example

1. Consider the function $g(x) = \cos^2(2\pi k_0 x)$. Sketch this function. You sample this signal in the spatial domain with a sampling rate $K_S = 1/\Delta x$ (e.g. samples spaced at intervals of Δx). What is the minimum sampling rate that you can use without aliasing? Give an intuitive explanation for your answer.

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Example

Assume that the Nyquist sampling periods of $f(x)$ and $g(x)$ are Δf and Δg, respectively. Determine the Nyquist sampling periods for

$a) f(x - x_0)$

$b) f(x) + g(x)$

$c) f(x) * f(x)$

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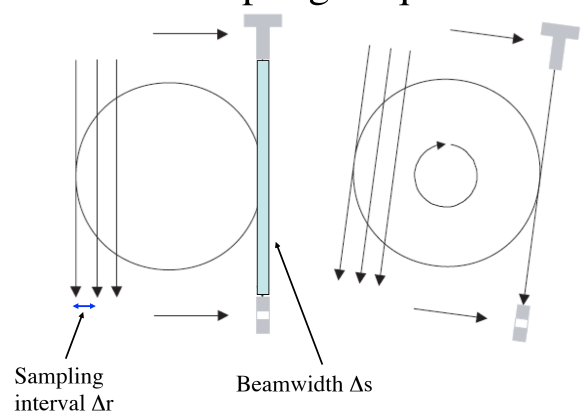
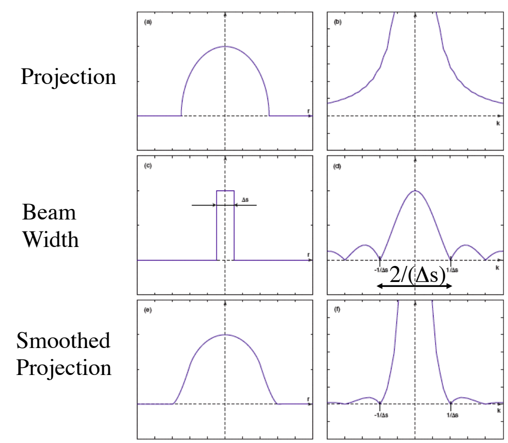
from Prince and Links 2006

Detector Sampling Requirements



Suetens 2002

Smoothing of Projection



Suetens 2002

Smoothing of Projection

$$g_s(l,\theta) = rect(l/\Delta s) * g(l,\theta)$$

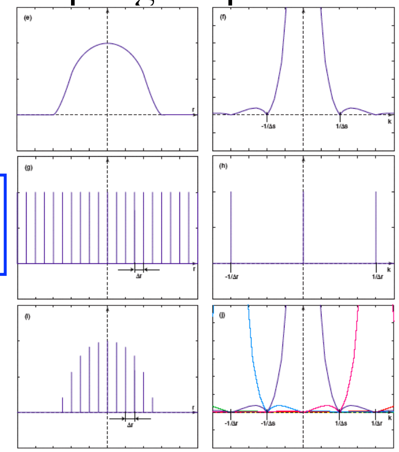
$$G_s(k_x,\theta) = \Delta s \, \mathrm{sinc}(k_x \Delta s) G(k_x,\theta)$$

Sampling Requirements

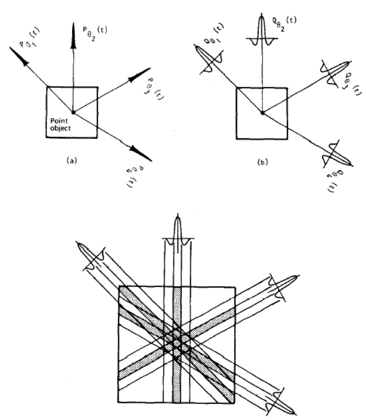
Smoothed Projection

Detectors $\Delta r \leq \Delta s/2$

Sampled Smooth Projection



View Aliasing



View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

$$\frac{\pi FOV}{N_{views}} = \Delta r$$

$$N_{views,360} = \frac{\pi FOV}{\Delta r} = \pi N_{proj} \quad \text{(for 360 degrees)}$$

$$N_{views,180} = \frac{\pi N_{proj}}{2} \quad \text{(for 180 degrees)}$$

Example

beamwidth $\Delta s = 1$ mm

Field of View (FOV) = 50 cm

$\Delta r = \Delta s/2 = 0.5$ mm

500 mm/ 0.5 mm = N = 1000 detector samples

$\pi * N = 3146$ views per 360 degrees

$\approx$ 1500 views per 180 degrees

CT "Rule of Thumb"

$$N_{view} = N_{\det ectors} = N_{pixels}$$

Suetens 2002

Example

Consider a rectangular object of width 20mm and height 40mm centered at (-10mm, -10mm). The attenuation coefficient of the object is 1 mm^{-1}. The object is imaged with a 1st generation CT scanner with a beamwidth of 1mm. The desired FOV is 100 mm.

Determine the appropriate detector size Δr and the number of radial samples needed to span the FOV. Assume that the middle two samples are acquired at coordinates of $-\Delta r/2$ and $\Delta r/2$.

Determine the number of angular samples required.

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