Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
MRI Lecture 2

2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi\left(k_x x + k_y y\right)}dxdy$$

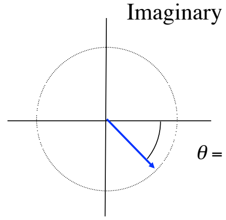
Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi\left(k_x x + k_y y\right)}dk_x dk_y$$

Phasor Diagram

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp\left(-j2\pi k_x x\right)dx$$

$\theta = -2\pi k_x x$

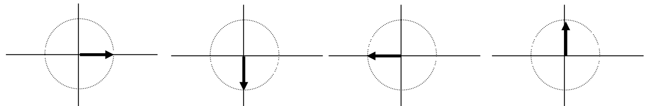


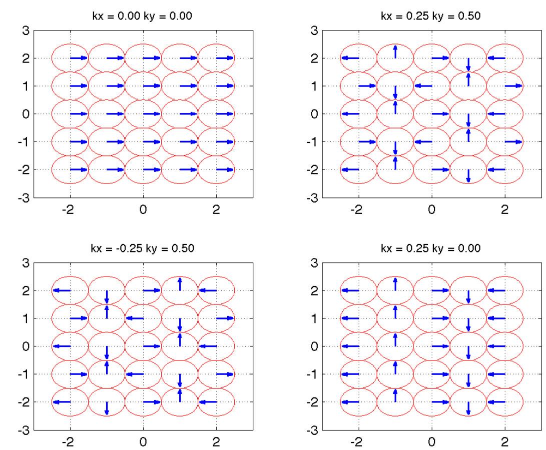
$k_x = 1;\ x = 0$ | $2\pi k_x x = 0$

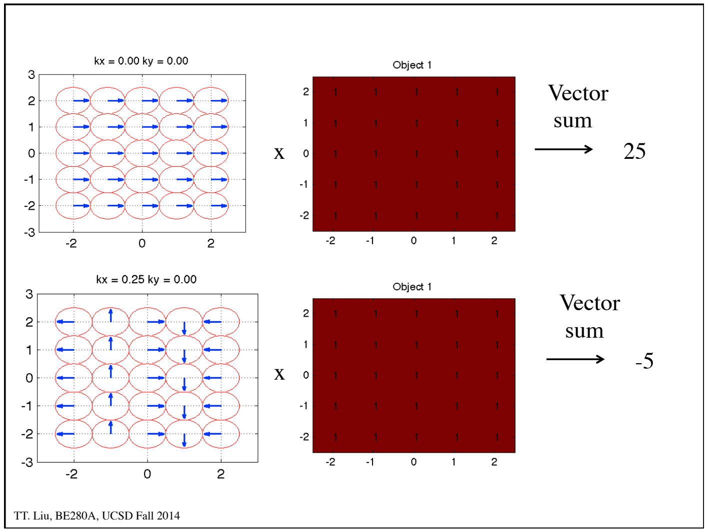
$x = 1/4$ | $2\pi k_x x = \pi/2$

$x = 1/2$ | $2\pi k_x x = \pi$

$x = 3/4$ | $2\pi k_x x = 3\pi/2$






kx = 0.00 ky = 0.00
Object 1
X
Vector sum
25
kx = 0.25 ky = 0.00
Object 1
X
Vector sum
-5
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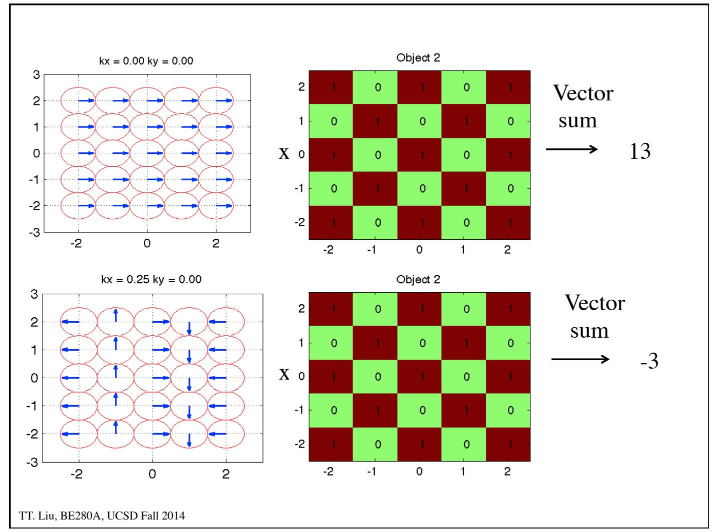
kx = 0.00 ky = 0.00
Object 2
X
Vector sum
13
kx = 0.25 ky = 0.00
Object 2
X
Vector sum
-3
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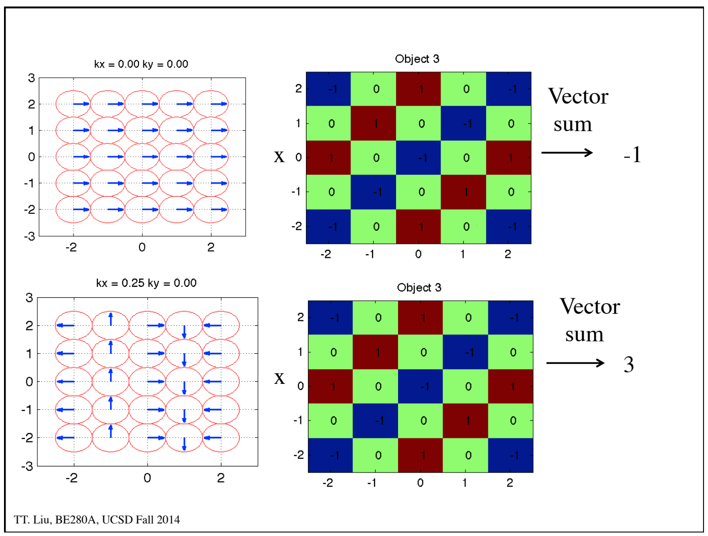
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Object 3
X
Vector sum
-1
kx = 0.25 ky = 0.00
Object 3
X
Vector sum
3
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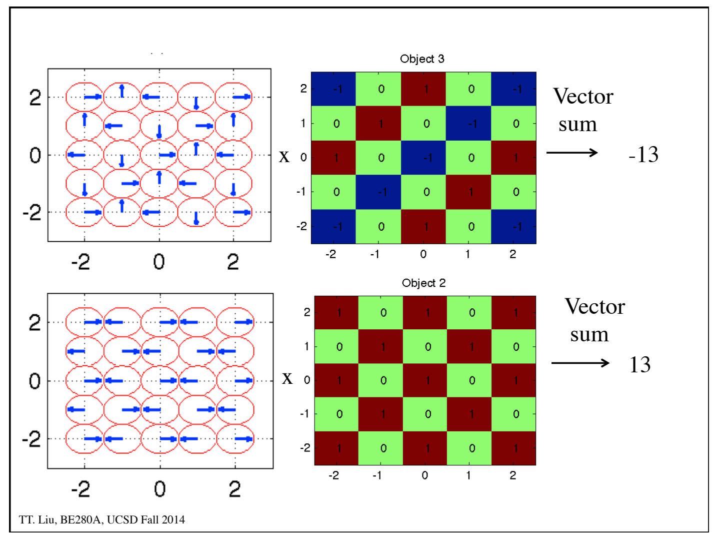
Object 3
X
Vector sum
-13
Object 2
X
Vector sum
13
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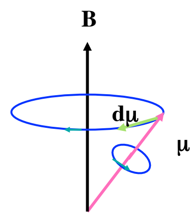
Precession

$$\frac{d\mu}{dt} = \mu \times \gamma \mathbf{B}$$



Analogous to motion of a gyroscope

Precesses at an angular frequency of

$$\omega = \gamma B$$

This is known as the **Larmor** frequency.

Larmor Frequency

$\omega = \gamma B$ Angular frequency in rad/sec

$f = \gamma B / (2\pi)$ Frequency in cycles/sec or Hertz, Abbreviated Hz

For a 1.5 T system, the Larmor frequency is 63.86 MHz which is 63.86 million cycles per second. For comparison, KPBS-FM transmits at 89.5 MHz.

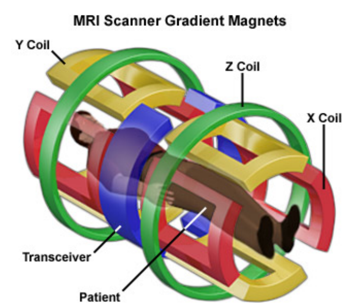
Note that the earth's magnetic field is about 50 μT, so that a 1.5T system is about 30,000 times stronger.

Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

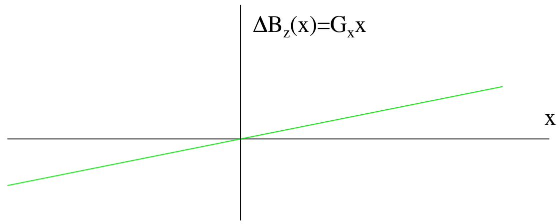
Gradient coils are used to add a spatial variation to B_z such that $B_z(x,y,z) = B_0 + \Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.

MRI Gradients



http://www.magnet.fsu.edu/education/tutorials/magnetacademy/mri/fullarticle.html
http://gulfnews.com/business/economy/german-factories-race-to-meet-big-surge-in-orders-1.638066

Interpretation



Spins Precess at $\gamma B_0 - \gamma G_x x$ (slower)

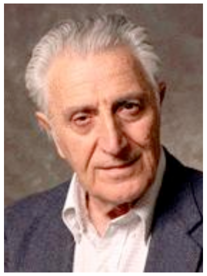
Spins Precess at γB_0

Spins Precess at at $\gamma B_0 + \gamma G_x x$ (faster)

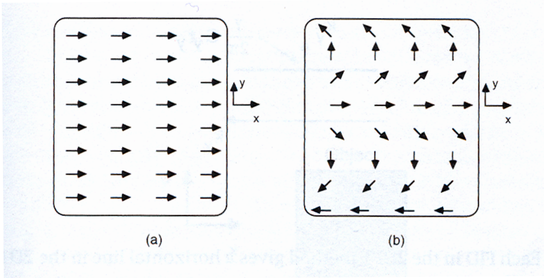
Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

Spins

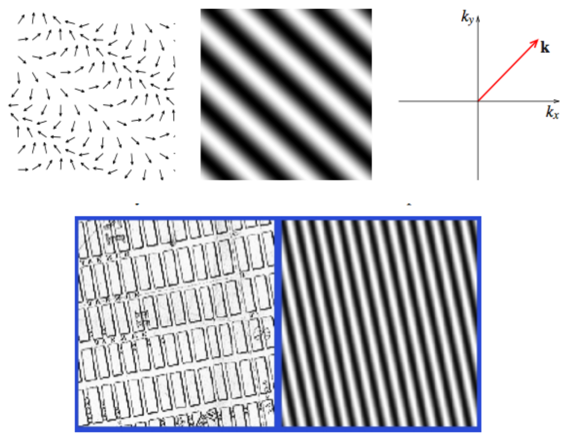


There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.
Erwin Hahn

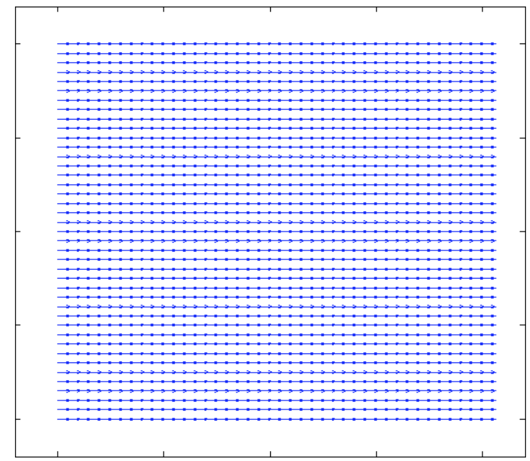


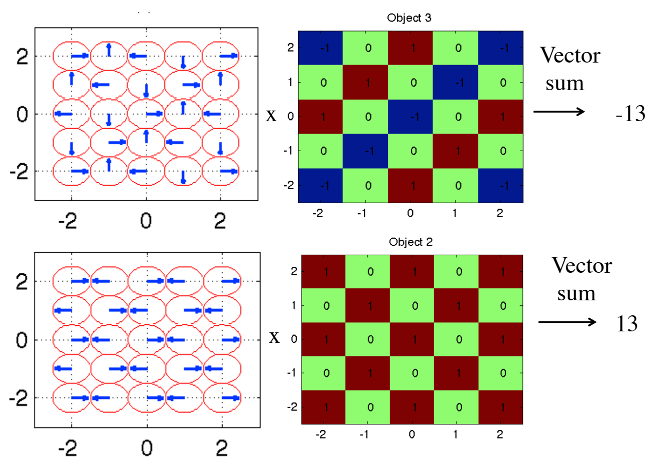
$k_x=0; k_y=0$ $k_x=0; k_y \neq 0$

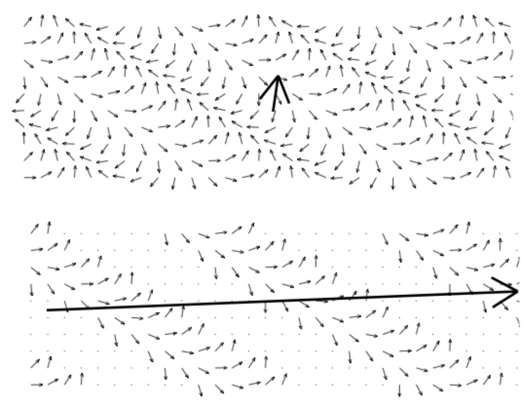
Fig 3.12 from Nishimura



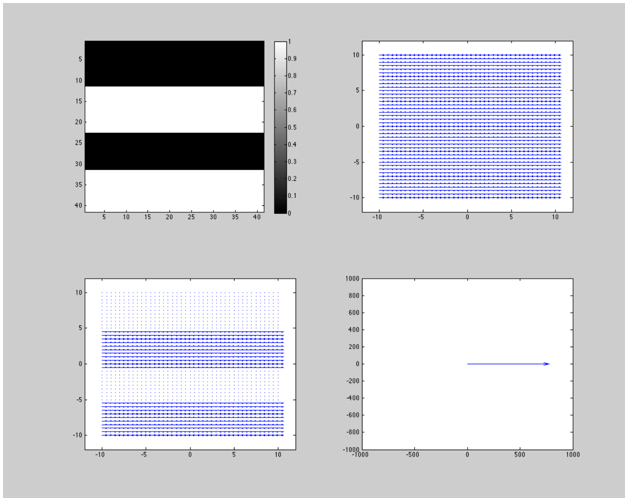
Hanson 2009

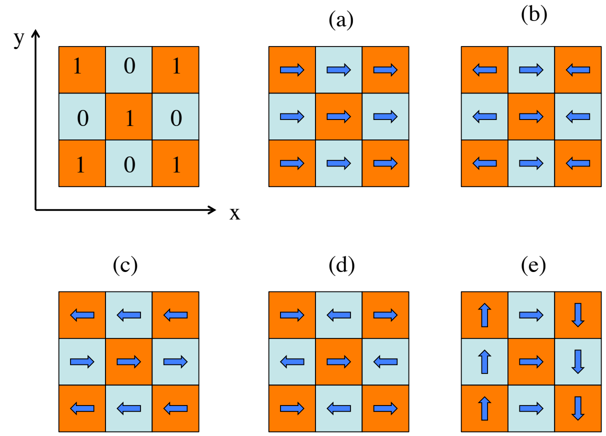






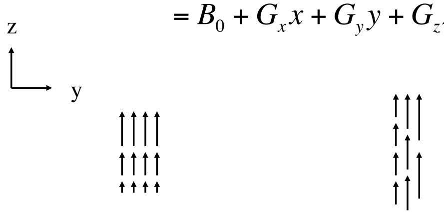
Hanson 2009





Gradient Fields

$$B_z(x,y,z) = B_0 + \frac{\partial B_z}{\partial x}x + \frac{\partial B_z}{\partial y}y + \frac{\partial B_z}{\partial z}z$$
$$= B_0 + G_x x + G_y y + G_z z$$



$$G_z = \frac{\partial B_z}{\partial z} > 0 \qquad G_y = \frac{\partial B_z}{\partial y} > 0$$

Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G}\cdot\vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t)\cdot\vec{r}$$

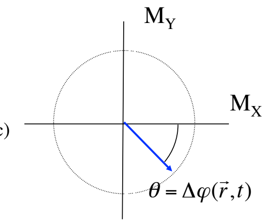
Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned}\omega(\vec{r},t) &= \gamma B_z(\vec{r},t) \\ &= \gamma B_0 + \gamma \vec{G}(t)\cdot\vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r},t)\end{aligned}$$

Phase

$$M(\vec{r},t) = M(\vec{r},0)e^{\varphi(\vec{r},t)}$$



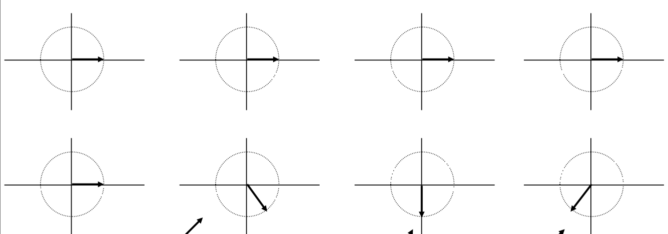
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\begin{aligned}\varphi(\vec{r},t) &= -\int_0^t \omega(\vec{r},\tau)d\tau \\ &= -\omega_0 t + \Delta\varphi(\vec{r},t)\end{aligned}$$

Where the incremental phase due to the gradients is

$$\begin{aligned}\Delta\varphi(\vec{r},t) &= -\int_0^t \Delta\omega(\vec{r},\tau)d\tau \\ &= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau\end{aligned}$$

Phase with constant gradient

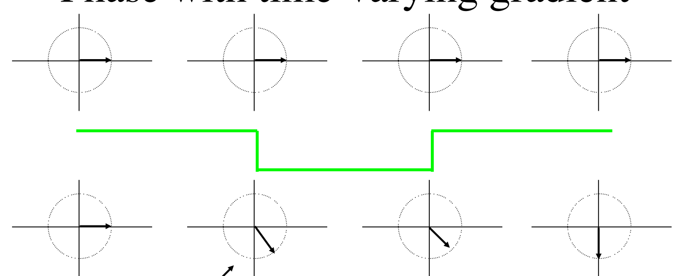


$$\begin{aligned}\Delta\varphi(\vec{r},t_1) &= -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\gamma G_x x t_1\end{aligned}$$

$$\begin{aligned}\Delta\varphi(\vec{r},t_2) &= -\int_0^{t_2}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\Delta\omega(\vec{r})t_2 \\ &= -\gamma G_x x t_2\end{aligned}$$

$$\begin{aligned}\Delta\varphi(\vec{r},t_3) &= -\int_0^{t_3}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\gamma G_x x t_3\end{aligned}$$

Phase with time-varying gradient



$$\begin{aligned}\Delta\varphi(\vec{r},t_1) &= -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\gamma G_x x t_1\end{aligned}$$

$$\begin{aligned}\Delta\varphi(\vec{r},t_2) &= -\int_0^{t_2}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau + 0 \\ &= -\gamma G_x x t_1\end{aligned}$$

$$\begin{aligned}\Delta\varphi(\vec{r},t_3) &= -\int_0^{t_3}\Delta\omega(\vec{r},\tau)d\tau \\ &= -\gamma G_x x\left(t_1 + (t_3 - t_2)\right)\end{aligned}$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$M(\vec{r},t) = M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)}$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right)$$
$$= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)$$

Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r},t)dV$$
$$= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$s(t) = e^{j\omega_0 t}s_r(t)$$
$$= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$
$$= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \left[G_x(\tau)x + G_y(\tau)y\right]d\tau\right)dxdy$$
$$= \int_x\int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$s(t) = \int_x\int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$
$$= M\left(k_x(t),k_y(t)\right)$$
$$= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M(k_x(t), k_y(t)) = F[m(x,y)]\big|_{k_x(t), k_y(t)}$$

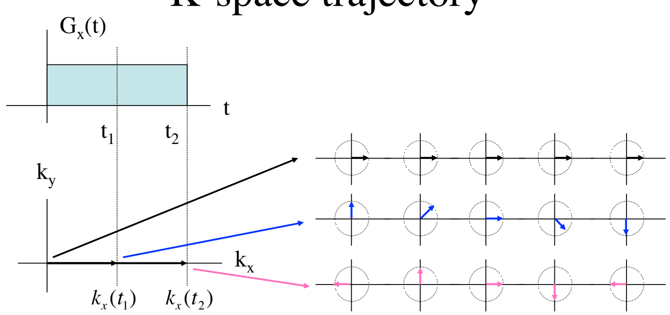
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

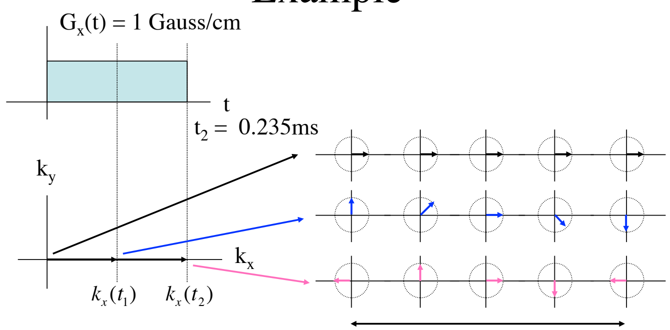
Gradient strengths have units of (magnetic field)/ distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

Example



$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)\,d\tau$$
$$= 4257\,Hz/G \cdot 1G/cm \cdot 0.235\times10^{-3}\,s$$
$$= 1\ cm^{-1}$$

K-space trajectory