Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
MRI Lecture 3

2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x+k_y y)}dxdy$$

Inverse Fourier Transform

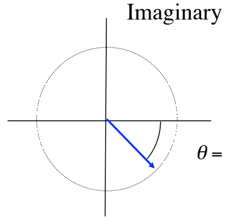
$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x+k_y y)}dk_x dk_y$$

Phasor Diagram

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$\theta = -2\pi k_x x$

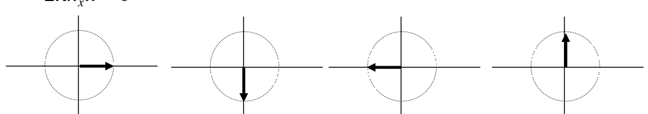
Imaginary



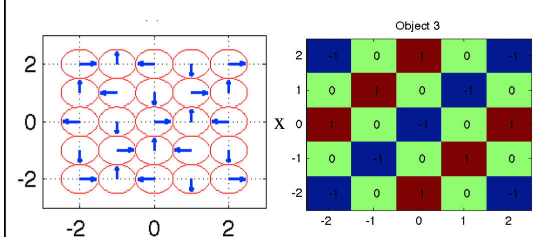
Real

$\theta = -2\pi k_x x$

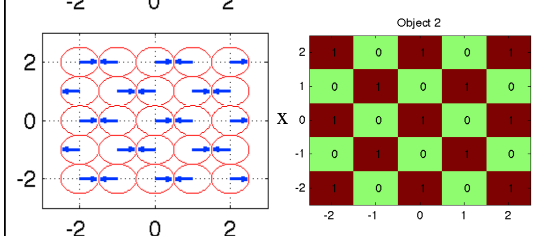
$k_x = 1;\ x = 0$	$x = 1/4$	$x = 1/2$	$x = 3/4$
$2\pi k_x x = 0$	$2\pi k_x x = \pi/2$	$2\pi k_x x = \pi$	$2\pi k_x x = 3\pi/2$



$\theta = 0$ $\quad \theta = -\pi/2$ $\quad \theta = -\pi$ $\quad \theta = -3\pi/2$



Vector sum → -13

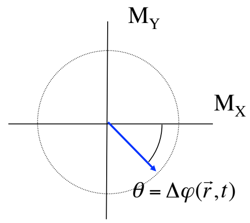


Vector sum → 13

Phase

$$M(\vec{r},t) = M(\vec{r},0)e^{\varphi(\vec{r},t)}$$

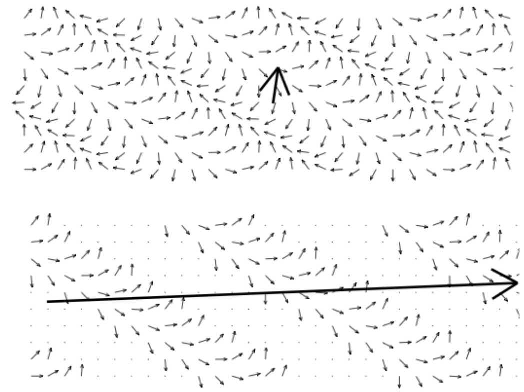
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency



$$\begin{aligned}\varphi(\vec{r},t) &= -\int_0^t \omega(\vec{r},\tau)d\tau \\ &= -\omega_0 t + \Delta\varphi(\vec{r},t)\end{aligned}$$

Where the incremental phase due to the gradients is

$$\begin{aligned}\Delta\varphi(\vec{r},t) &= -\int_0^t \Delta\omega(\vec{r},\tau)d\tau \\ &= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau\end{aligned}$$



Hanson 2009

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$\begin{aligned}M(\vec{r},t) &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)} \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right) \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)\end{aligned}$$

Signal Equation

Signal from a volume

$$\begin{aligned}s_r(t) &= \int_V M(\vec{r},t)dV \\ &= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz\end{aligned}$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$
\begin{aligned}
s(t) &= e^{j\omega_0 t} s_r(t) \\
&= \int_x \int_y m(x,y)\exp\left(-j\gamma \int_o^t \vec{G}(\tau)\cdot \vec{r}\, d\tau\right) dxdy \\
&= \int_x \int_y m(x,y)\exp\left(-j\gamma \int_o^t \left[G_x(\tau)x + G_y(\tau)y\right] d\tau\right) dxdy \\
&= \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy
\end{aligned}
$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$
\begin{aligned}
s(t) &= \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy \\
&= M\left(k_x(t), k_y(t)\right) \\
&= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}
\end{aligned}
$$

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

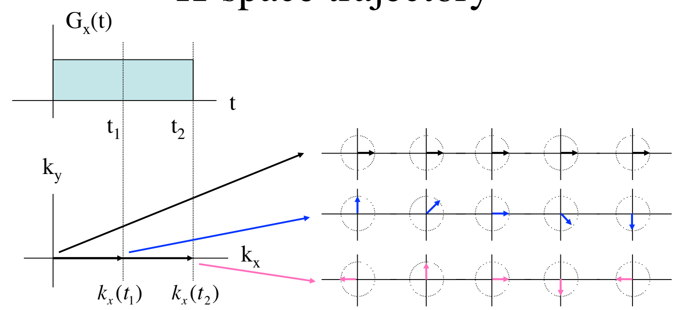
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

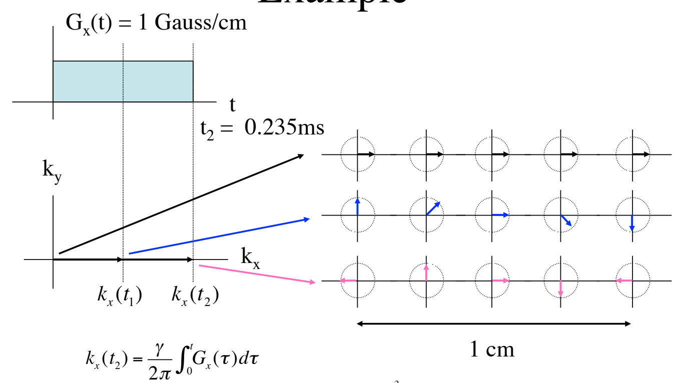
Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/ distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

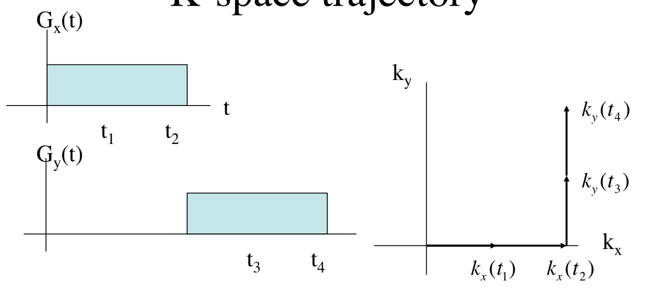
$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

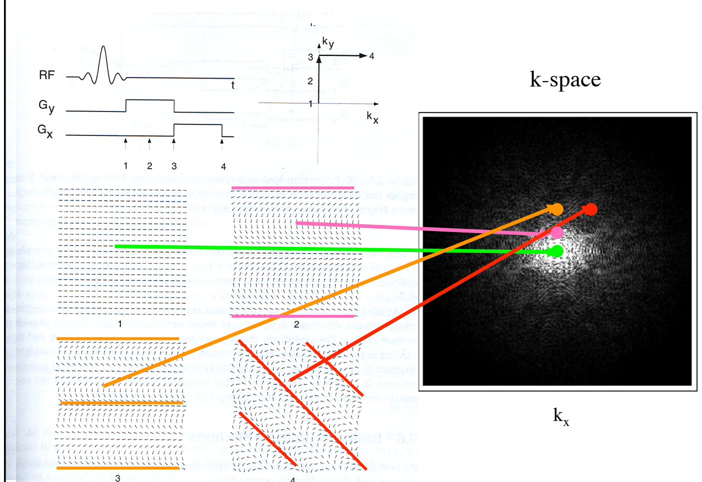
Example

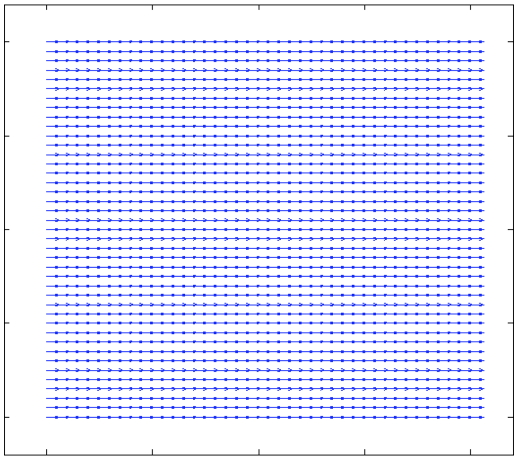


$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= 4257 Hz/G \cdot 1G/cm \cdot 0.235\times10^{-3} s$$
$$= 1\ cm^{-1}$$

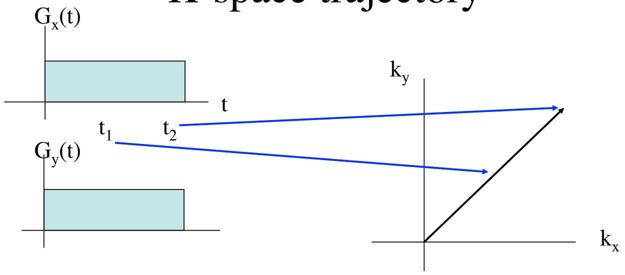
K-space trajectory



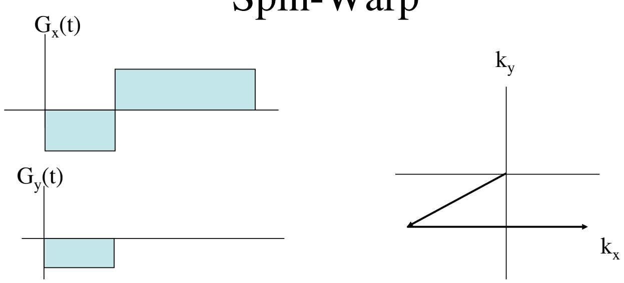




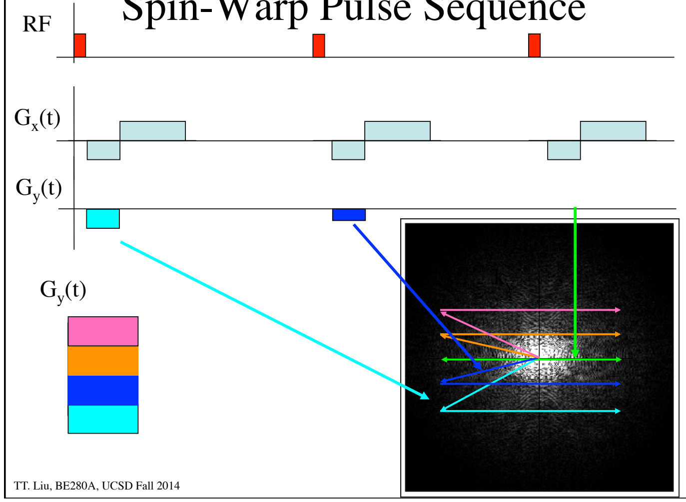
K-space trajectory

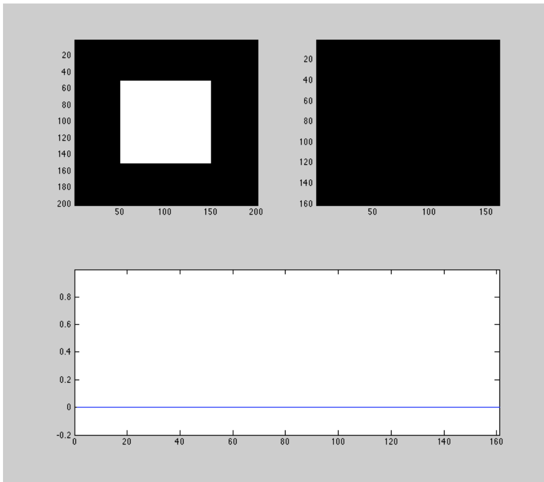


Spin-Warp

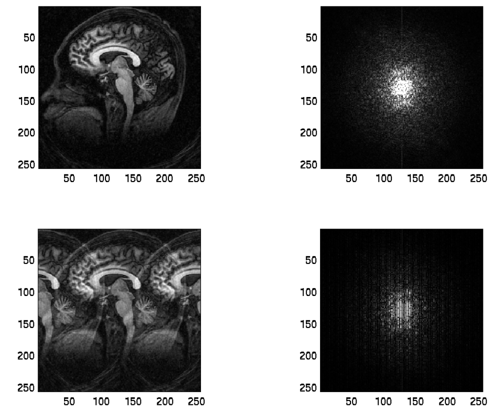


Spin-Warp Pulse Sequence

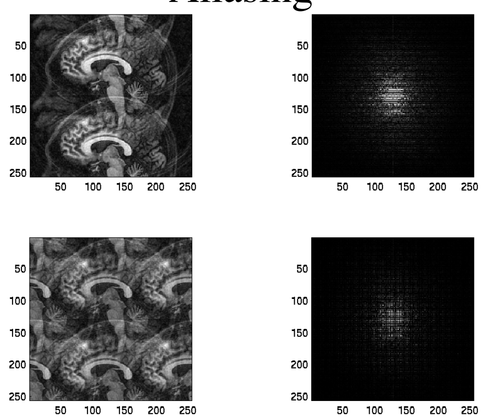




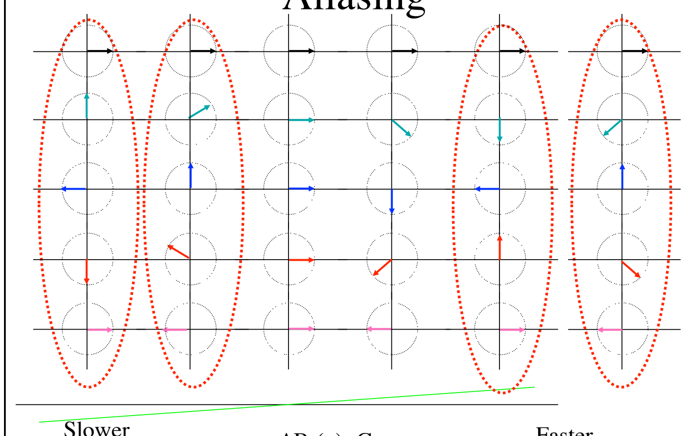
Sampling in k-space



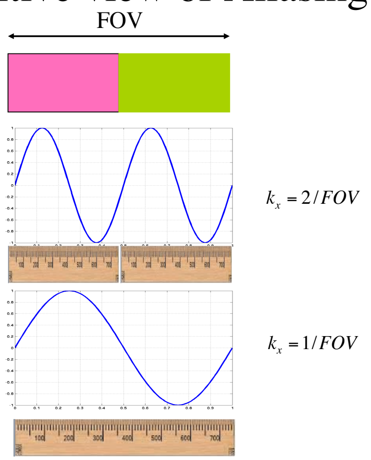
Aliasing



Aliasing



Intuitive view of Aliasing



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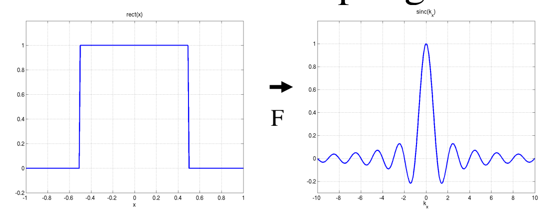
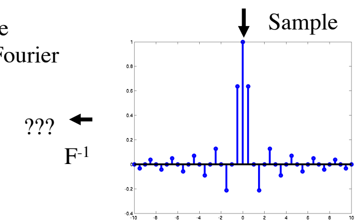
Fourier Sampling



Instead of sampling the signal, we sample its Fourier Transform



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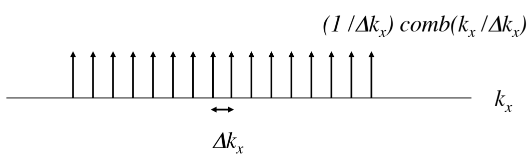
Fourier Sampling



$$G_S(k_x) = G(k_x)\frac{1}{\Delta k_x}comb\left(\frac{k_x}{\Delta k_x}\right)$$

$$= G(k_x)\sum_{n=-\infty}^{\infty}\delta(k_x - n\Delta k_x)$$

$$= \sum_{n=-\infty}^{\infty}G(n\Delta k_x)\delta(k_x - n\Delta k_x)$$

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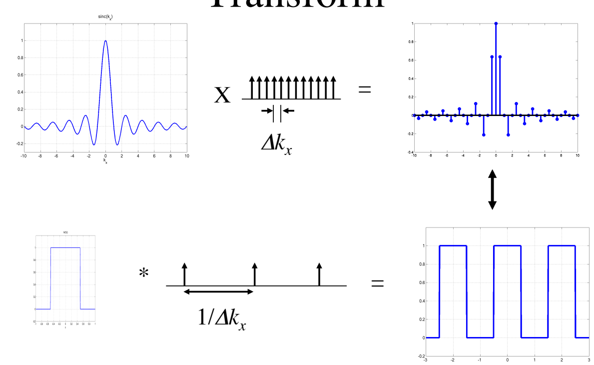
Fourier Sampling -- Inverse Transform



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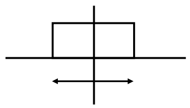
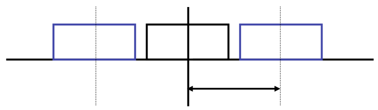
Nyquist Condition



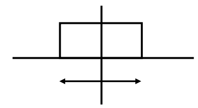
FOV (Field of View)



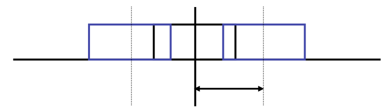
$1/\Delta k_x$

To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/\text{FOV}$

Aliasing



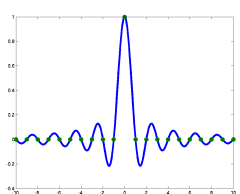
FOV (Field of View)

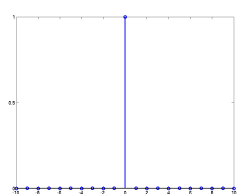


$1/\Delta k_x$

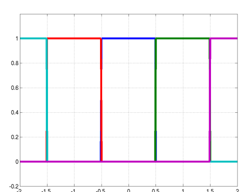
Aliasing occurs when $1/\Delta k_x <$ FOV

Aliasing Example





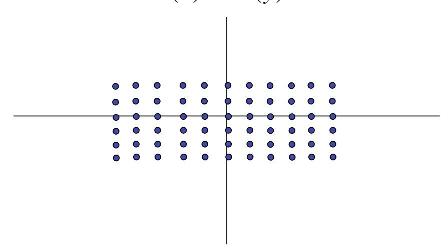
$\Delta k_x = 1$



$1/\Delta k_x = 1$

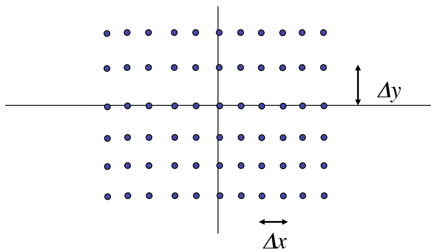
2D Comb Function

$$\begin{aligned} comb(x,y) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m, y-n) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m)\delta(y-n) \\ &= comb(x)comb(y) \end{aligned}$$



Scaled 2D Comb Function

$$comb(x/\Delta x, y/\Delta y) = comb(x/\Delta x)comb(y/\Delta y)$$
$$= \Delta x \Delta y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - m\Delta x)\delta(y - n\Delta y)$$



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2D k-space sampling

$$G_S(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)$$
$$= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}G(m\Delta k_x, n\Delta k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

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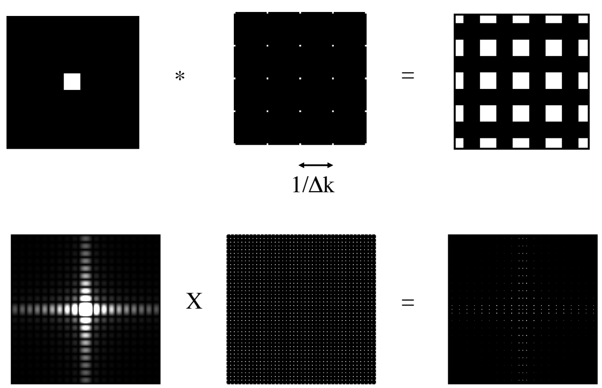

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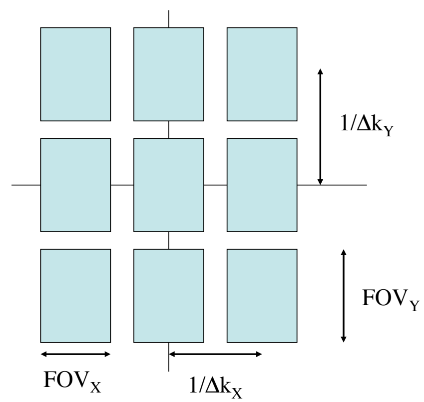
Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

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Windowing

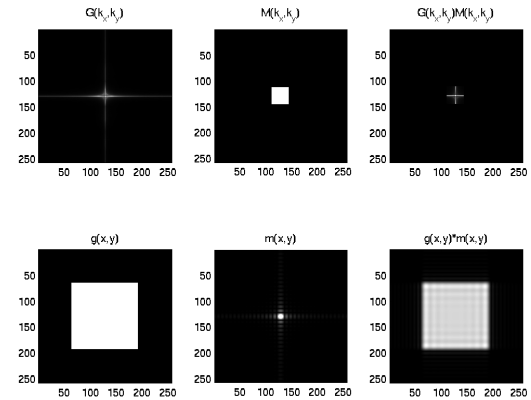
Windowing the data in Fourier space

$$G_W(k_x,k_y) = G(k_x,k_y)W(k_x,k_y)$$

Results in convolution of the object with the inverse transform of the window

$$g_W(x,y) = g(x,y) * w(x,y)$$

Resolution



Windowing Example

$$W(k_x,k_y) = rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)$$

$$w(x,y) = F^{-1}\left[rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)\right]$$

$$= W_{k_x}W_{k_y}\,\mathrm{sinc}(W_{k_x}x)\,\mathrm{sinc}(W_{k_y}y)$$

$$g_W(x,y) = g(x,y) ** W_{k_x}W_{k_y}\,\mathrm{sinc}(W_{k_x}x)\,\mathrm{sinc}(W_{k_y}y)$$

Effective Width

$$w_E = \frac{1}{w(0)}\int_{-\infty}^{\infty} w(x)dx$$

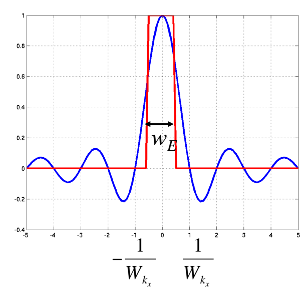
Example

$$w_E = \frac{1}{1}\int_{-\infty}^{\infty}\mathrm{sinc}(W_{k_x}x)dx$$

$$= F\left[\mathrm{sinc}(W_{k_x}x)\right]\Big|_{k_x=0}$$

$$= \frac{1}{W_{k_x}}rect\left(\frac{k_x}{W_{k_x}}\right)\Bigg|_{k_x=0}$$

$$= \frac{1}{W_{k_x}}$$

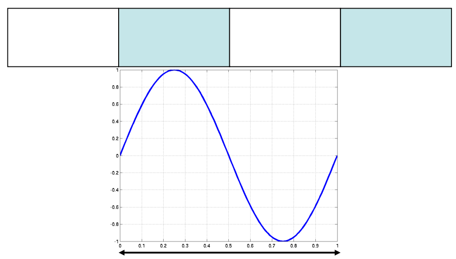


Resolution and spatial frequency

With a window of width W_{k_x} the highest spatial frequency is $W_{k_x}/2$.

This corresponds to a spatial period of $2/W_{k_x}$.

$\frac{1}{W_{k_x}}$ = Effective Width



$\frac{2}{W_{k_x}}$

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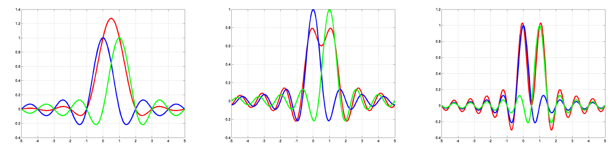
Windowing Example

$$g(x,y) = [\delta(x) + \delta(x-1)]\delta(y)$$

$$\begin{aligned} g_W(x,y) &= [\delta(x) + \delta(x-1)]\delta(y) ** W_{k_x} W_{k_y} \operatorname{sinc}(W_{k_x} x)\operatorname{sinc}(W_{k_y} y) \\ &= W_{k_x} W_{k_y} \left([\delta(x) + \delta(x-1)] * \operatorname{sinc}(W_{k_x} x)\right)\operatorname{sinc}(W_{k_y} y) \\ &= W_{k_x} W_{k_y} \left(\operatorname{sinc}(W_{k_x} x) + \operatorname{sinc}(W_{k_x}(x-1))\right)\operatorname{sinc}(W_{k_y} y) \end{aligned}$$



$W_{k_x} = 1$ $\quad W_{k_x} = 1.5$ $\quad W_{k_x} = 2$

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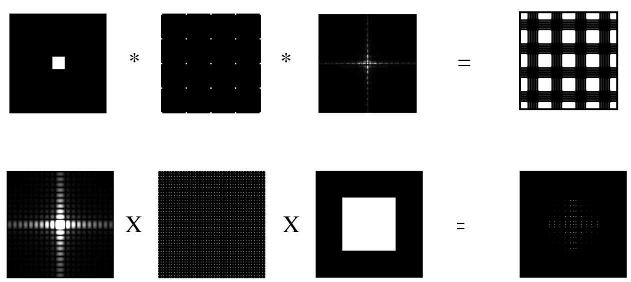
Sampling and Windowing



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Sampling and Windowing

Sampling and windowing the data in Fourier space

$$G_{SW}(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Results in replication and convolution in object space.

$$g_{SW}(x,y) = W_{k_x} W_{ky} g(x,y) ** comb(\Delta k_x x, \Delta k_y y) ** \operatorname{sin}c(W_{k_x} x)\operatorname{sin}c(W_{k_y} y)$$

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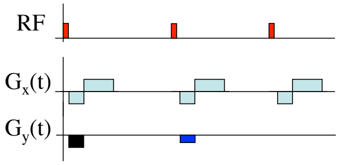
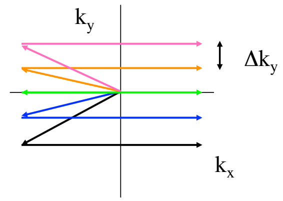
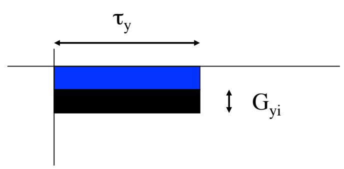
Sampling in k_y







$$\Delta k_y = \frac{\gamma}{2\pi} G_{yi} \tau_y$$

$$FOV_y = \frac{1}{\Delta k_y}$$

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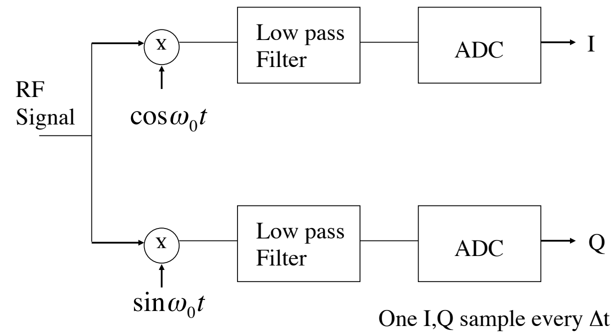
Sampling in k_x



One I,Q sample every Δt

M= I+jQ

Note: In practice, there are number of ways of implementing this processing.

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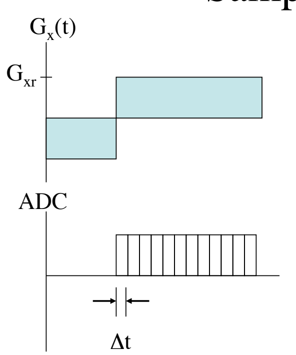
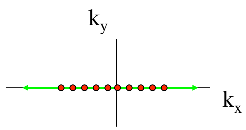
Sampling in k_x





$$\Delta k_x = \frac{\gamma}{2\pi} G_{xr} \Delta t$$

$$FOV_x = \frac{1}{\Delta k_x}$$

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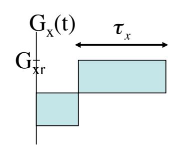
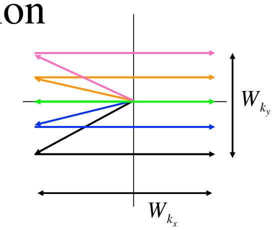
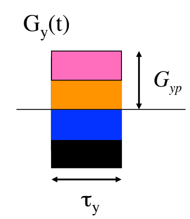
Resolution

$$\delta_x = \frac{1}{W_{k_x}} = \frac{1}{2k_{x,\max}} = \frac{1}{\frac{\gamma}{2\pi} G_{xr} \tau_x}$$





$$\delta_y = \frac{1}{W_{k_y}} = \frac{1}{2k_{y,\max}} = \frac{1}{\frac{\gamma}{2\pi} 2 G_{yp} \tau_y}$$



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Example

Goal:

$FOV_x = FOV_y = 25.6$ cm

$\delta_x = \delta_y = 0.1$ cm

Readout Gradient:

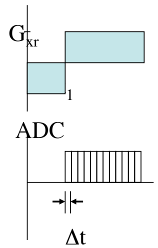
$$FOV_x = \frac{1}{\frac{\gamma}{2\pi} G_{xr} \Delta t}$$

Pick $\Delta t = 32\ \mu$sec

$$G_{xr} = \frac{1}{FOV_x \frac{\gamma}{2\pi} \Delta t} = \frac{1}{(25.6cm)(42.57\times 10^6 T^{-1}s^{-1})(32\times 10^{-6} s)}$$

$$= 2.8675\ \times 10^{-5}\text{T/cm}$$
$$= .28675\ \text{G/cm}$$

1 Gauss $= 1\times 10^{-4}$ Tesla



Example

Readout Gradient:

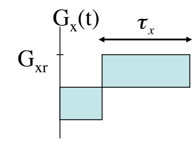
$$\delta_x = \frac{1}{\frac{\gamma}{2\pi} G_{xr} \tau_x}$$

$$\tau_x = \frac{1}{\delta_x \frac{\gamma}{2\pi} G_{xr}} = \frac{1}{(0.1cm)(4257\ G^{-1}s^{-1})(0.28675\ \text{G/cm})}$$

$$= 8.192\ \text{ms}$$
$$= N_{read}\Delta t$$

where

$$N_{read} = \frac{FOV_x}{\delta_x} = 256$$



Example

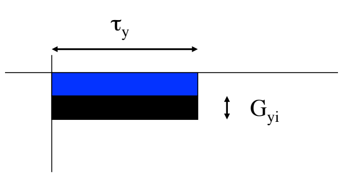
Phase - Encode Gradient:

$$FOV_y = \frac{1}{\frac{\gamma}{2\pi} G_{yi} \tau_y}$$

Pick $\tau_y = 4.096\ m$sec

$$G_{yi} = \frac{1}{FOV_y \frac{\gamma}{2\pi} \tau_y} = \frac{1}{(25.6cm)(42.57\times 10^6 T^{-1}s^{-1})(4.096\times 10^{-3} s)}$$

$$= 2.2402\times 10^{-7}\ \text{T/cm}$$
$$= .00224\ \text{G/cm}$$



Example

Phase - Encode Gradient:

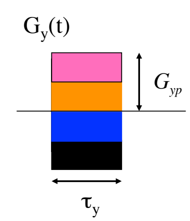
$$\delta_y = \frac{1}{\frac{\gamma}{2\pi} 2 G_{yp} \tau_y}$$

$$G_{yp} = \frac{1}{\delta_y 2 \frac{\gamma}{2\pi} \tau_y} = \frac{1}{(0.1cm)(4257\ G^{-1}s^{-1})(4.096\times 10^{-3} s)}$$

$$= 0.2868\ \text{G/cm}$$
$$= \frac{N_p}{2} G_{yi}$$

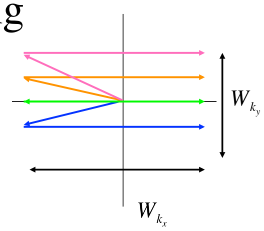
where

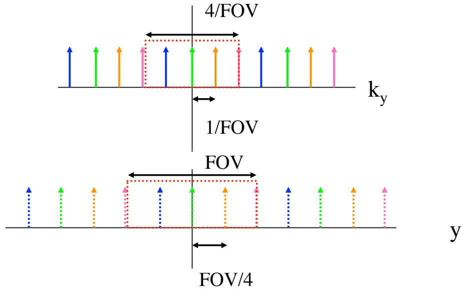
$$N_p = \frac{FOV_y}{\delta_y} = 256$$



Sampling

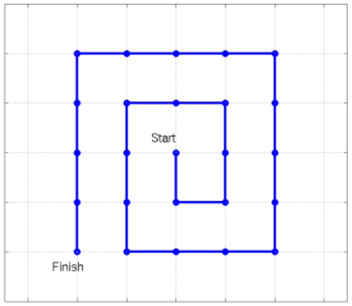
In practice, an even number (typically power of 2) sample is usually taken in each direction to take advantage of the Fast Fourier Transform (FFT) for reconstruction.





Example

Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with $\Delta t = 10\ \mu\text{sec}$. The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly.



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