Longitudinal Relaxation

\[ \frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1} \]

After a 90 degree pulse

\[ M_z(t) = M_0(1 - e^{-t/T_1}) \]

Due to exchange of energy between nuclei and the lattice (thermal vibrations). Process continues until thermal equilibrium as determined by Boltzmann statistics is obtained.

The energy \( \Delta E \) required for transitions between down to up spins, increases with field strength, so that \( T_1 \) increases with \( B \).

T2 Relaxation

\[ M_{xy}(t) = M_0 e^{-t/T_2} \]

After a 90 degree excitation

Summary

1) Longitudinal component recovers exponentially.

2) Transverse component precesses and decays exponentially.

Fact: Can show that \( T_2 < T_1 \) in order for \( |M(t)| \leq M_0 \)

Physically, the mechanisms that give rise to \( T_1 \) relaxation also contribute to transverse \( T_2 \) relaxation.
Signal Decay

Some inhomogeneity, Some dephasing

More inhomogeneity, More dephasing, Decrease in MR signal

\[ \exp \left( -\frac{t}{T_2^*} \right) \]

The overall decay has the form.

Due to random motions of spins. Not reversible.

Due to static inhomogeneities. Reversible with a spin-echo sequence.

Spin Echo

Discovered by Erwin Hahn in 1950.

The spin-echo can refocus the dephasing of spins due to static inhomogeneities. However, there will still be $T_2^*$ dephasing due to random motion of spins.

There is nothing that nuclear spins will not do for you, as long as you treat them as human beings. Erwin Hahn
Spin Echo

Phase at time $\tau$

$$\varphi(\tau) = \int_{0}^{\tau} -\omega_E(\vec{r}) dt = -\omega_E(\vec{r}) \tau$$

Phase after 180 pulse

$$\varphi(\tau^+) = \omega_E(\vec{r}) \tau$$

Phase at time $2\tau$

$$\varphi(2\tau) = -\omega_E(\vec{r}) \tau + \omega_E(\vec{r}) \tau = 0$$
Image Contrast

Different tissues exhibit different relaxation rates, $T_1$, $T_2$, and $T_2^*$. In addition different tissues can have different densities of protons. By adjusting the pulse sequence, we can create contrast between the tissues. The most basic way of creating contrast is adjusting the two sequence parameters: TE (echo time) and TR (repetition time).

Saturation Recovery Sequence

Gradient Echo

$\begin{align*}
I(x,y) &= \rho(x,y) \left[ 1 - e^{-TR/T_1(x,y)} \right] e^{-TE/T_2^*(x,y)}
\end{align*}$

Spin Echo

$\begin{align*}
I(x,y) &= \rho(x,y) \left[ 1 - e^{-TR/T_1(x,y)} \right] e^{-TE/T_2(x,y)}
\end{align*}$

T1-Weighted Scans

Make TE very short compared to either $T_2$ or $T_2^*$. The resultant image has both proton and $T_1$ weighting.

$\begin{align*}
I(x,y) &\approx \rho(x,y) \left[ 1 - e^{-TR/T_1(x,y)} \right]
\end{align*}$
T2-Weighted Scans

Make TR very long compared to $T_1$ and use a spin-echo pulse sequence. The resultant image has both proton and $T_2$ weighting.

$$I(x,y) \approx \rho(x,y)e^{-TE/T_2}$$

Proton Density Weighted Scans

Make TR very long compared to $T_1$ and use a very short TE. The resultant image is proton density weighted.

$$I(x,y) \approx \rho(x,y)$$

Example

<table>
<thead>
<tr>
<th>T1-weighted</th>
<th>Density-weighted</th>
<th>T2-weighted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tissue</td>
<td>Proton Density</td>
<td>T1 (ms)</td>
</tr>
<tr>
<td>CSF</td>
<td>1.0</td>
<td>6000</td>
</tr>
<tr>
<td>Gray</td>
<td>0.85</td>
<td>1350</td>
</tr>
<tr>
<td>White</td>
<td>0.7</td>
<td>850</td>
</tr>
</tbody>
</table>
FLASH sequence

\[ I(x,y) = \rho(x,y) \left[ \frac{1 - e^{-TR/T_1(x,y)}}{1 - e^{-TR/T_2(x,y)}} \right] \frac{\sin \theta}{\cos \theta} \exp(-TE/T_2) \]

Signal intensity is maximized at the Ernst Angle

\[ \theta_E = \cos^{-1}(\exp(-TR/T_1)) \]

FLASH equation assumes no coherence from shot to shot. In practice this is achieved with RF spoiling.

\[ \theta_E = \cos^{-1}(\exp(-TR/T_1)) \]
Inversion Recovery

$I(x, y) = \rho(x, y) \left[1 - 2e^{-TR/T_1(x, y)} + e^{-TR/T_1(x, y)}\right] e^{-TE/T_2(x, y)}$

Intensity is zero when inversion time is

$TI = -T_1 \ln \left[\frac{1 + \exp(-TR/T_1)}{2}\right]$
RF Excitation

At equilibrium, net magnetization is parallel to the main magnetic field. How do we tip the magnetization away from equilibrium?

\[ B_1 \]

radiofrequency field tuned to Larmor frequency and applied in transverse \((xy)\) plane induces nutation (at Larmor frequency) of magnetization vector as it tips away from the z-axis. - lab frame of reference

Image & caption: Nishimura, Fig. 3.2

http://www.eecs.umich.edu/~dnol/BME516/
Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

\[ B_1(t) = 2B(t)\cos(\omega t)i \]
\[ = B(t)\cos(\omega t)i - \sin(\omega t)j + B(t)\cos(\omega t)i + \sin(\omega t)j \]
Precession

Analogous to motion of a gyroscope
Precesses at an angular frequency of

\[
\omega = \gamma B
\]

This is known as the Larmor frequency.

Rotating Frame Bloch Equation

\[
\frac{dM_{rot}}{dt} = M_{rot} \times \gamma B_{eff}
\]

\[
B_{eff} = B_{rot} + \omega_{rot} \gamma = \begin{bmatrix} 0 \\ 0 \\ -\omega \end{bmatrix}
\]

Note: we use the RF frequency to define the rotating frame. If this RF frequency is on-resonance, then the main B0 field doesn’t cause any precession in the rotating frame. However, if the RF frequency is off-resonance, then there will be a net precession in the rotating frame that is given by the difference between the RF frequency and the local Larmor frequency.

Let \( B_{rot} = B_1(t) \hat{i} + B_j \hat{k} \)

\[
B_{eff} = B_{rot} + \frac{\omega_{rot}}{\gamma} = B_1(t) \hat{i} + \left( B_0 - \frac{\omega}{\gamma} \right) \hat{k}
\]

If \( \omega = \omega_0 = \gamma B_0 \)

Then \( B_{eff} = B_1(t) \hat{i} \)
Flip angle

\[ \theta = \int_0^\tau \omega_1(s)ds \]

where

\[ \omega_1(t) = \gamma B_1(t) \]

Example

\[ \tau = 400 \, \mu\text{sec}; \quad \theta = \pi/2 \]

\[ B_1 = \frac{\theta}{\gamma \tau} = \frac{\pi/2}{2\pi(4257 \, \text{Hz}/G)(400e - 6)} = 0.1468 \, G \]
Let \( \mathbf{B}_{rot} = B_1(t)\mathbf{i} + (B_0 + \gamma G_z)\mathbf{k} \)

\[
\mathbf{B}_{eff} = \mathbf{B}_{rot} + \frac{\omega}{\gamma} = B_1(t)\mathbf{i} + \left( B_0 + \gamma G_z - \frac{\omega}{\gamma} \right)\mathbf{k}
\]

If \( \omega = \omega_0 \)

\[
\mathbf{B}_{eff} = B_1(t)\mathbf{i} + (\gamma G_z)\mathbf{k}
\]

Slice Selection

\[
W = \gamma G \Delta z / (2\pi)
\]

Sinc function

Small Tip Angle Approximation

For small \( \theta \)

\[
M_z = M_0 \cos \theta = M_0
\]

\[
M_{xy} = M_0 \sin \theta \approx M_0 \theta
\]
Excitation k-space

At each time increment of width $\Delta \tau$, the excitation $B_i(\tau)$ produces an increment in magnetization of the form $\Delta M_\omega = jM_\omega \theta(\tau) = jM_\omega \gamma B_i(\tau) \Delta \tau$ (small tip angle approximation).

Excitation k-space

In the presence of a gradient, this will accumulate phase of the form $\varphi = \gamma \int G_z(s) ds$, such that the incremental magnetization at time $t$ is

\[ \Delta M_\omega(t, z : \tau) = jM_\omega \gamma B_i(\tau) \exp\left( -j \int G_z(s) ds \right) \Delta \tau \]

Integrating over all time increments, we obtain

\[ M_\omega(t, z) = jM_\omega \gamma B_i(\tau) \exp\left( -j \int G_z(s) ds \right) d\tau \]

where $k(\tau, t) = \frac{\gamma}{2\pi} \int G_z(s) ds$

This has the form of a Fourier transform, where we are integrating the contributions of the field $B_i(\tau)$ at the k-space point $k(\tau, t)$.

Excitation k-space

\[ k(\tau, t) = \frac{\gamma}{2\pi} \int_\tau^t G_z(s) ds \]

\[ M_\omega \theta \exp(-j2\pi k(\tau, t)z) \]

Excitation k-space

\[ \Delta \omega \tau \]

\[ \theta(\tau) = \gamma B_i(\tau) \Delta \tau \]

\[ jM_\omega \gamma B_i(\tau) \Delta \tau \]
**Excitation k-space**

$$M_{xy}(t,z) = j\gamma M_0 \int \gamma B_z(\tau) \exp(-j2\pi k(\tau,t)z)d\tau$$

This has the form of a Fourier transform, where we are integrating the contributions of the field $B_z(\tau)$ at the k-space point $k(\tau,t)$.

---

**Refocusing**

$$M_{xy}(t,z) = j\gamma M_0 \int \gamma B_z(\tau) \exp(-j2\pi k(\tau,t)z)d\tau$$

This has the form of a Fourier transform, where we are integrating the contributions of the field $B_z(\tau)$ at the k-space point $k(\tau,t)$.

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**Slice Selection**

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**Gradient Echo**
Slice Selection

\[ \Delta f = \frac{1}{\tau} = \frac{\Delta z}{2\pi} \cdot \text{sinc}(t/\tau) \]