Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

Note that in the rotating frame, there is no precession about the z-axis. From the point of view of the precession of the spins, this is equivalent to the z-component of the field being set equal to zero.
**Rotating Frame Bloch Equation**

\[
\frac{dM_{rot}}{dt} = M_{rot} \times \gamma B_{\text{eff}}
\]

\[
B_{\text{eff}} = B_{\text{rot}} + \frac{\omega_{\text{rot}}}{\gamma}; \quad \omega_{\text{rot}} = \begin{bmatrix} 0 \\ 0 \\ -\omega \end{bmatrix}
\]

Note: we use the RF frequency to define the rotating frame. If this RF frequency is on-resonance, then the main B0 field doesn’t cause any precession in the rotating frame. However, if the RF frequency is off-resonance, then there will be a net precession in the rotating frame that is given by the difference between the RF frequency and the local Larmor frequency.

**Flip angle**

\[
\theta = \int_0^S \omega_{1}(s)ds
\]

where

\[
\omega_{1}(t) = \gamma B_{1}(t)
\]

**Example**

\[
\tau = 400 \ \mu\text{sec}; \ \theta=\frac{\pi}{2}
\]

\[
B_{1} = \frac{\theta}{\gamma \tau} = \frac{\pi/2}{2\pi(4257 \text{ Hz} / G)(400e^{-6})} = 0.1468 \ G
\]
Let \( \mathbf{B}_{\text{rot}} = B_1(t)i + (B_0 + \gamma G_z)\mathbf{k} \)

\[
\mathbf{B}_{\text{eff}} = \mathbf{B}_{\text{rot}} + \frac{\omega_{\text{rot}}}{\gamma} \\
= B_1(t)i + \left( B_0 + \frac{\gamma G_z}{\gamma} \right)\mathbf{k}
\]

If \( \omega = \omega_0 \)

\[
\mathbf{B}_{\text{eff}} = B_1(t)i + (\gamma G_z)\mathbf{k}
\]
Slice Selection

$W = \gamma G \Delta z (2\pi)$

Sinc($Wt$)

Small Tip Angle Approximation

For small $\theta$

$M_z = M_0 \cos \theta \approx M_0$

$M_{xy} = M_0 \sin \theta \approx M_0 \theta$

Excitation k-space

At each time increment of width $\Delta \tau$, the excitation $B_i(\tau)$ produces an increment in magnetization of the form

$\Delta M_{xy} = jM_0 \theta(\tau) = jM_0 \gamma B_i(\tau) \Delta \tau$

(small tip angle approximation)
**Excitation k-space**

In the presence of a gradient, this will accumulate phase of the form
\[ \varphi = \gamma \int_{t}^{\tau} z G_{i}(s) ds, \]

such that the incremental magnetization at time \( t \) is
\[ \Delta M_{o}(t, z ; \tau) = j M_{o} \gamma B_{i}(\tau) \exp \left( -j \gamma \int_{t}^{\tau} z G_{i}(s) ds \right) \Delta \tau \]

This has the form of a Fourier transform, where we are integrating the contributions of the field \( B_{i}(\tau) \) at the k-space point \( k(\tau, t) \).

**Excitation k-space**

Integrating over all time increments, we obtain
\[ M_{o}(t, z) = j M_{o} \int_{t}^{\tau} \gamma B_{i}(\tau) \exp \left( -j \gamma \int_{t}^{\tau} z G_{i}(s) ds \right) d\tau \]
\[ = j M_{o} \int_{t}^{\tau} \gamma B_{i}(\tau) \exp \left( -j 2 \pi k(\tau, t) z \right) d\tau \]

This has the form of a Fourier transform, where we are integrating the contributions of the field \( B_{i}(\tau) \) at the k-space point \( k(\tau, t) \).

**Refocusing**

\[ M_{o}(t, z) = j M_{o} \int_{t}^{\tau} \gamma B_{i}(\tau) \exp \left( -j 2 \pi k(\tau, t) z \right) d\tau \]

This has the form of a Fourier transform, where we are integrating the contributions of the field \( B_{i}(\tau) \) at the k-space point \( k(\tau, t) \).
Slice Selection

Slice select gradient
Slice refocusing gradient

Gradient Echo

Slice select gradient
Slice refocusing gradient

Spins all in phase at $k_{x=0}$

$\Delta f = \frac{1}{\tau} = \frac{\gamma G_z \Delta z}{2\pi}$

Nishimura 1996
Multi-dimensional Excitation $k$-space

$$M_x(t,r) = jM_0 \int_{-\infty}^{t} \omega(t) \exp(-jy\int_{s}^{t} G(s) \cdot rds)d\tau$$

$$= jM_0 \int_{-\infty}^{t} \omega(t) \exp(j2\pi k(\tau) \cdot r)d\tau$$

where

$$k(\tau) = -\frac{\gamma}{2\pi} \int_{0}^{\tau} G(t')dt'$$

Pauly et al 1989

Exercise: Sketch the quiver diagrams corresponding to the contributions of the two RF pulses and show that they produce the desired pattern.

Pauly et al 1989
So far we have assumed that the spins are not moving (aside from thermal motion giving rise to relaxation), and contrast has been based upon $T_1$, $T_2$, and proton density. We were able to achieve different contrasts by adjusting the appropriate pulse sequence parameters.

Biological samples are filled with moving spins, and we can also use MRI to image the movement. Examples: blood flow, diffusion of water in the white matter tracts. In addition, we can also sometimes induce motion into the object to image its mechanical properties, e.g. imaging of stress and strain with MR elastography.

\[ \phi(t) = -\int_0^t \Delta \omega(\tau) d\tau \]
\[ = -\int_0^t \gamma \Delta B(\tau) d\tau \]
\[ = -\int_0^t \gamma \mathbf{G}(\tau) \cdot \mathbf{v}(\tau) d\tau \]
\[ = -\gamma \int_0^t [G_x(\tau)x(\tau) + G_y(\tau)y(\tau) + G_z(\tau)z(\tau)] d\tau \]
Phase of Moving Spin

Consider motion along the x-axis

\[ x(t) = x_0 + vt + \frac{1}{2} at^2 \]

\[ q(t) = -\gamma \int_0^t G_x(\tau)x(\tau)d\tau \]

\[ = -\gamma \int_0^t G_x(\tau)\left[x_0 + vt + \frac{1}{2} a\tau^2\right]d\tau \]

\[ = -\gamma \left[x_0 \int_0^t G_x(\tau)d\tau + \frac{3}{2} \int_0^t G_x(\tau)\tau^2d\tau + \frac{a}{2} \int_0^t G_x(\tau)d\tau \right] \]

\[ = -\gamma \left[ x_0 M_0 + vM_1 + \frac{a}{2} M_2 \right] \]

Flow Moment Example

\[ G_0 \]

\[ -G_0 \]

\[ M_0 = \int_0^T G_x(\tau)d\tau = 0 \]

\[ M_1 = \int_0^T G_x(\tau)\tau d\tau \]

\[ = -\int_0^T G_x d\tau + \int_0^T \frac{\tau^2}{2} G_x d\tau \]

\[ = G_0 \left[ \tau^2 \left( \frac{1}{2} \right) + \tau \left( \frac{1}{2} \right) \right] \]

\[ = G_0 \left[ \frac{T^2}{2} + \frac{T}{2} \right] = G_0 \frac{T^2}{2} \]

Phase Contrast Angiography (PCA)

\[ q_1 = -\gamma v M_1 = \gamma v G_0 T^2 \]

\[ q_2 = -\gamma v M_1 = -\gamma v G_0 T^2 \]

\[ \Delta \phi = q_1 - q_2 = 2\gamma v G_0 T^2 \]

\[ v_x = \frac{\Delta \phi}{2G_0 T^2} \]
**PCA example**

![PCA example image](http://www.medical.philips.com/main/products/mri/assets/images/case_of_week/cotw_51_s5.jpg)

**Diffusion**

- **2D random walk**
- \( N \) random steps of length \( d \)

\[
\langle \Delta x^2 \rangle = N d^2 = 2DT
\]

\( D = \) diffusivity

In brain:
\( D \approx 0.001 \text{ mm}^2/\text{s} \)

For \( T=100 \text{ msec} \),
\( \Delta x \approx 15 \mu \)

Credit: Larry Frank

**Diffusing Spins**

\[ \Delta B_z(x) \]

\[ \Delta B_z(x) \times \text{time} \]

**Diffusion Weighting**

\[
S \propto e^{-\gamma^2 G_0^2 \delta^2 DT} = e^{-bD} \quad \text{where} \quad b = \gamma^2 G_0^2 \delta^2 (T - \delta/3)
\]

**Diffusivity**
After a stroke, normal water movement is restricted in the region of damage. Diffusivity decreases, so the signal intensity increases.

Restricted Diffusion

D depends on direction

Diffusion tensor:
- 3 values of D
- 3 angles

Diffusion Imaging Example

Diffusion MRI Tractography from the Human Connectome Project