Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2015
CT/Fourier Lecture 2



# In-class Exercise

$$\mu(x,y) = rect(x, y/2)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

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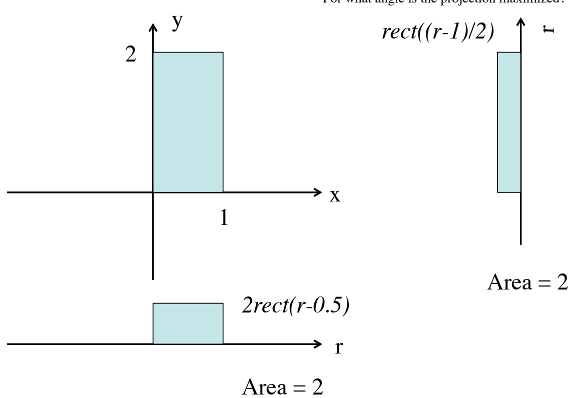
# Example

$$\mu(x,y) = rect(x-0.5,(y-1)/2)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?



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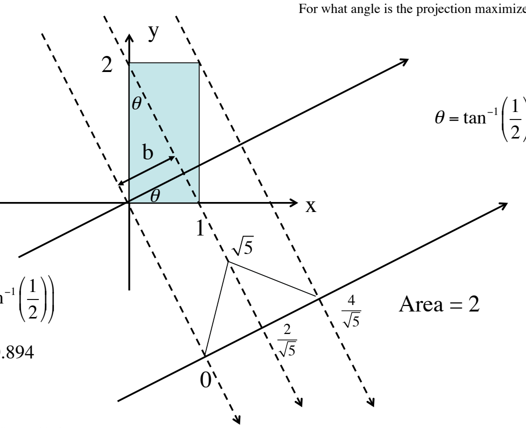



$$b = \cos\theta = \cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{2}{\sqrt{5}} \approx 0.894$$

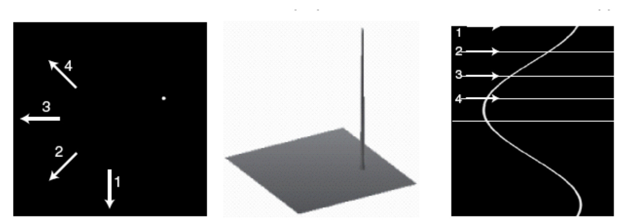
# Example

$$f(x,y) = \begin{cases} 1 & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} g(l,\theta=0) &= \int_{-\infty}^{\infty} f(l,y)\,dy \\ &= \int_{-\sqrt{1-l^2}}^{\sqrt{1-l^2}} dy \\ &= \begin{cases} 2\sqrt{1-l^2} & |l| \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$



# Sinogram

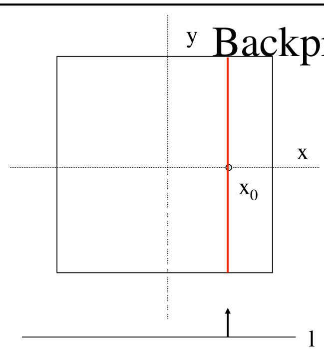



(a) (b) (c)



Suetens 2002

# Backprojection




$$\begin{aligned} b(x_0,y) &= p(l,\theta=0)\Delta\theta \\ &= p(x_0,0)\Delta\theta \end{aligned}$$

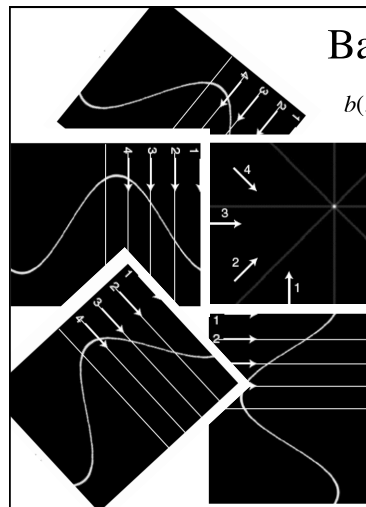
$$b_\theta(x,y) = g(x\cos\theta + y\sin\theta,\theta)\Delta\theta$$

$$\begin{aligned} b(x,y) &= B\{g(l,\theta)\} \\ &= \int_0^\pi g(x\cos\theta + y\sin\theta,\theta)\,d\theta \end{aligned}$$



Suetens 2002

# Backprojection

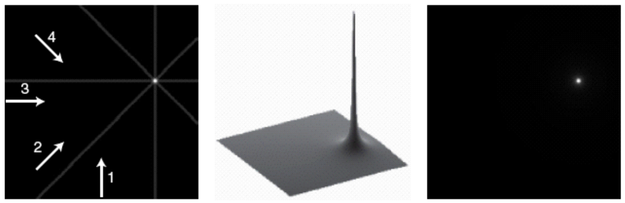


$$\begin{aligned} b(x,y) &= B\{p(l,\theta)\} \\ &= \int_0^\pi p(x\cos\theta + y\sin\theta,\theta)\,d\theta \end{aligned}$$



Suetens 2002

# Backprojection




$$b(x,y) = B\{p(l,\theta)\} = \int_0^{\pi} p(x\cos\theta + y\sin\theta, \theta)\, d\theta$$



Suetens 2002

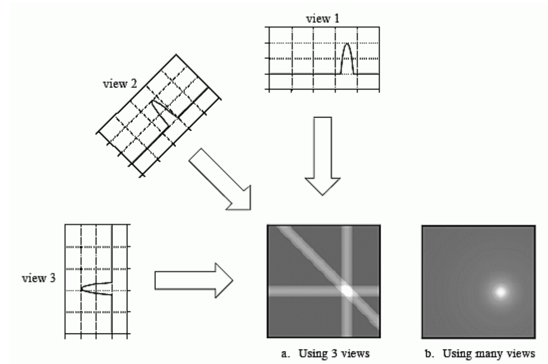



FIGURE 25-16
Backprojection. Backprojection reconstructs an image by taking each view and *smearing* it along the path it was originally acquired. The resulting image is a blurry version of the correct image.

http://www.dspguide.com/ch25/5.htm



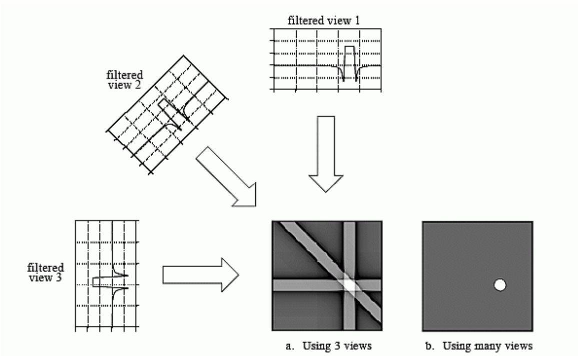
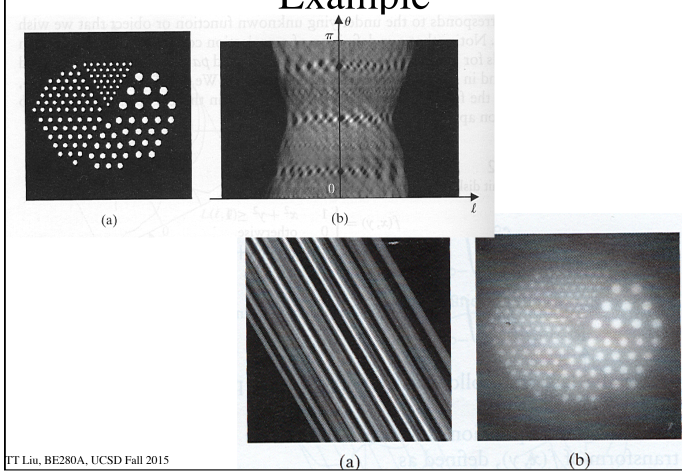



FIGURE 25-17
Filtered backprojection. Filtered backprojection reconstructs an image by filtering each view before backprojection. This removes the blurring seen in simple backprojection, and results in a mathematically exact reconstruction of the image. Filtered backprojection is the most commonly used algorithm for computed tomography systems.

http://www.dspguide.com/ch25/5.htm



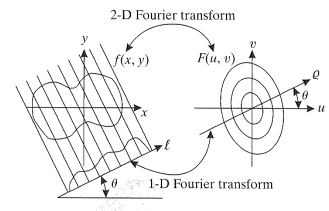
# Example





# Projection Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy$$
$$= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}$$

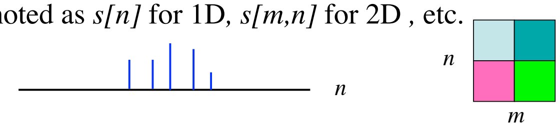




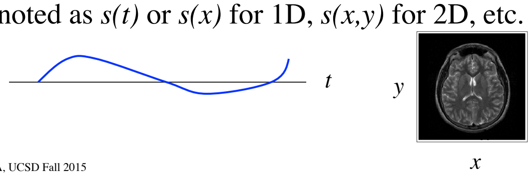

Prince&Links 2006

# Signals and Images

Discrete-time/space signal /image: continuous valued function with a discrete time/space index, denoted as $s[n]$ for 1D, $s[m,n]$ for 2D , etc.

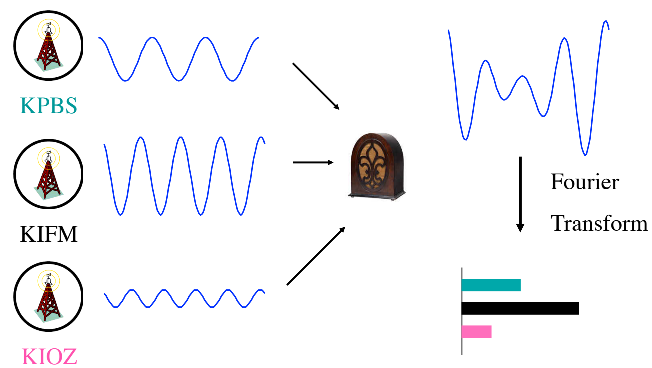



Continuous-time/space signal / image: continuous valued function with a continuous time/space index, denoted as $s(t)$ or $s(x)$ for 1D, $s(x,y)$ for 2D, etc.






# 1D Fourier Transform



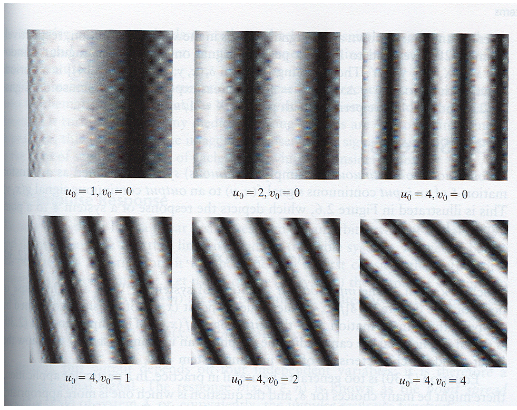
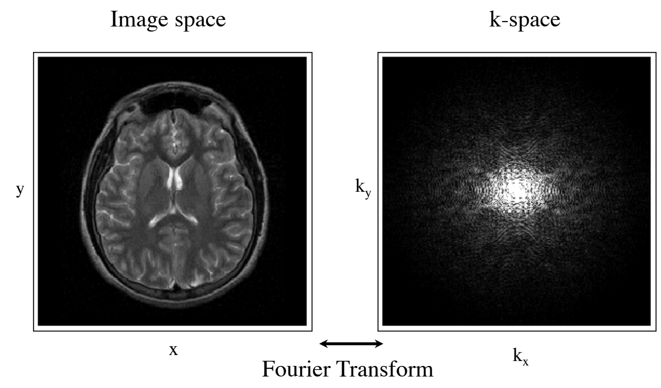


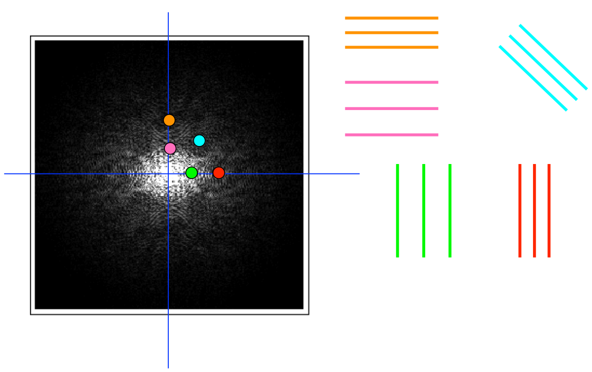

Figure 2.5 from Prince and Link



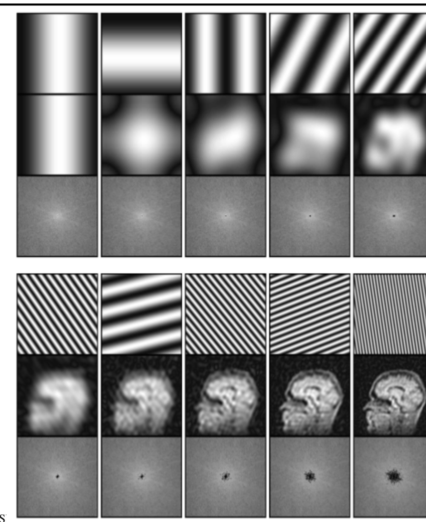
# k-space



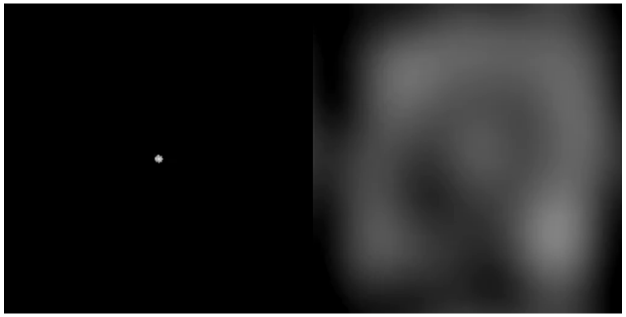




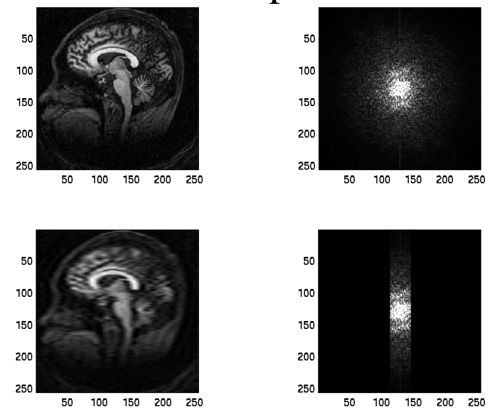


Hanson 2009



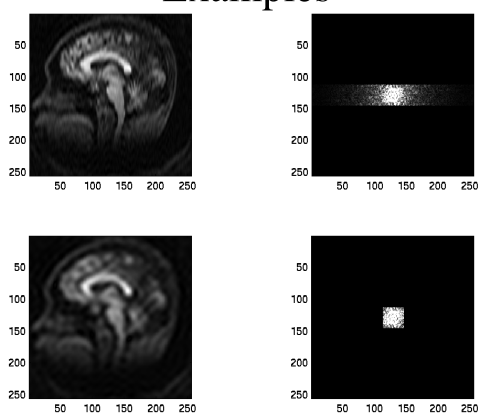


# Examples



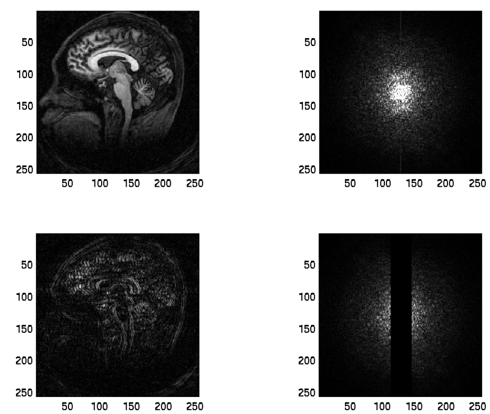


# Examples

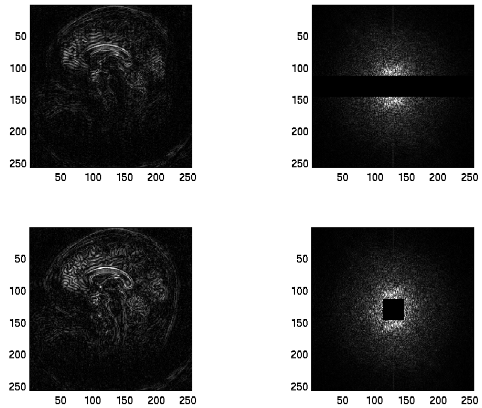




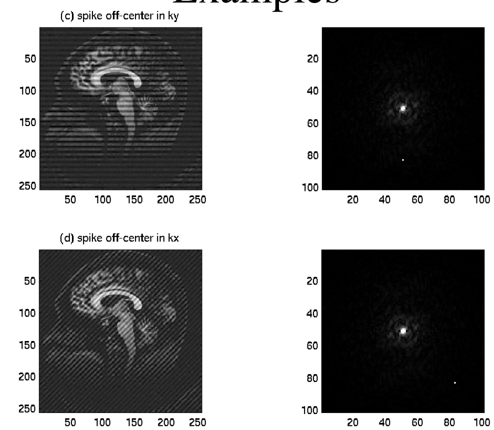
# Examples



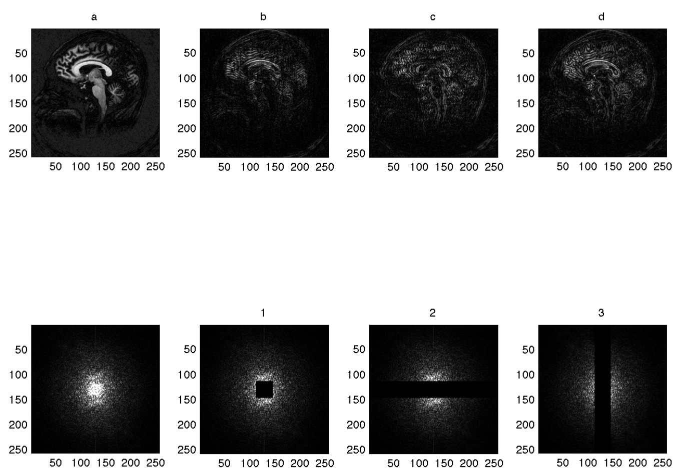
# Examples



# Examples



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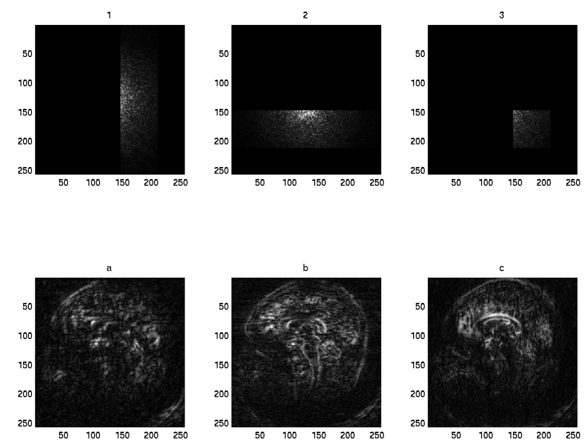
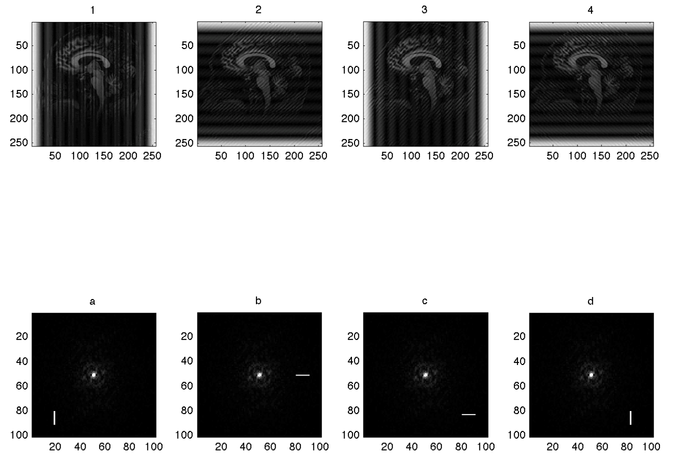



PollEv.com/be280a

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# The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df = F^{-1}\{G(f)\}$$



# Units

Temporal Coordinates, e.g. *t* in seconds, *f* in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. $x$ in cm, $k_x$ is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi k_x x}dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x)e^{j2\pi k_x x}dk_x \qquad \text{Inverse Fourier Transform}$$



# Complex Numbers

$j = \sqrt{-1}$

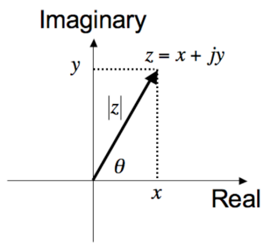
$j^2 = ?$

$(3+2j)(3-2j) = ?$

$j^2 = -1$

$(3+2j)(3-2j) = 9-4j^2 = 13$

# Complex Numbers




$$z = 2 + 1j$$

$$|z| = \sqrt{2^2 + 1} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.6 \text{ degrees}$$

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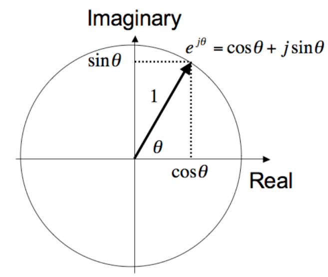
# Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy = |z|e^{j\theta}$$

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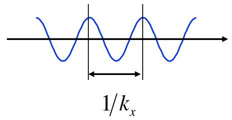
# 1D Fourier Transform

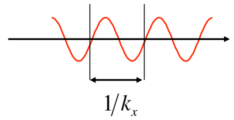
$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$

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# Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi k_x x}dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0)e^{-j2\pi k_x x}dx = e^{-j2\pi k_x x_0}$$

$$F(\Pi(x)) = \int_{-1/2}^{1/2} e^{-j2\pi k_x x}dx$$

$$= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x}$$

$$= \frac{\sin(\pi k_x)}{\pi k_x} = \mathrm{sinc}(k_x)$$

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## Computing Transforms

$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned}\int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x &= \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx \\ &= \int_{-\infty}^{\infty} g(-x) dx \\ &= G(0)\end{aligned}$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$



## Computing Transforms

Similarly,

$$F\{e^{j2\pi k_0 x}\} = \delta(k_x - k_0)$$
$$F\{\cos 2\pi k_0 x\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$
$$F\{\sin 2\pi k_0 x\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$



## Duality

Note the similarity between these two transforms

$$F\{e^{j2\pi ax}\} = \delta(k_x - a)$$
$$F\{\delta(x - a)\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\{G(x)\} = g(-k_x)$$



## Application of Duality

$$F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\{\Pi(x)\} = \text{sinc}(k_x)$.

Therefore from duality, $F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x)$

# 2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

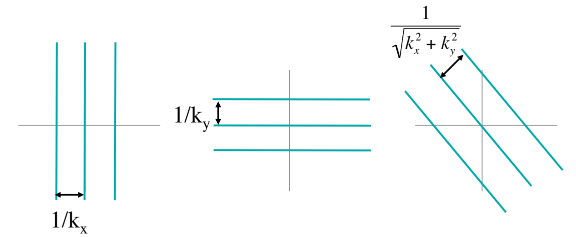
Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$



# Plane Waves

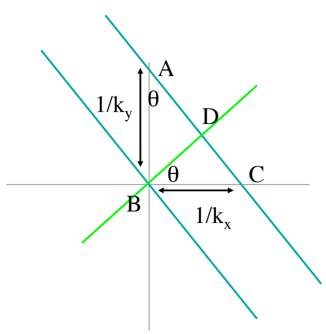
$$e^{j2\pi(k_x x + k_y y)} = \cos(2\pi(k_x x + k_y y)) + j\sin(2\pi(k_x x + k_y y))$$




cos(2πk$_x$x) cos(2πk$_y$y) cos(2πk$_x$x +2πk$_y$y)



# Plane Waves



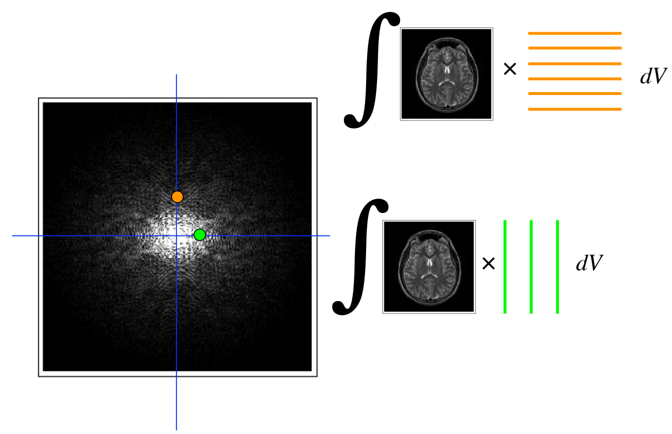

$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$





# Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$

The Fourier Transform is then separable as well.

$$G(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx\int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy$$
$$= G_X(k_x)G_Y(k_y)$$

*Example*

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

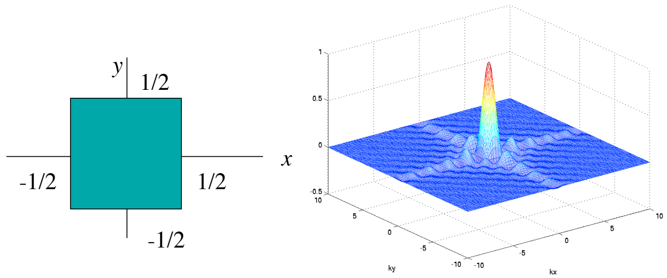


# Example (sinc/rect)

*Example*

$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$






# Examples

Is this function separable?
What is its Fourier Transform?

$$g(x,y) = \exp(-j2\pi(8x+9y))\sin(28\pi x)$$

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# Examples

$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$

$G(k_x,k_y) = 1$

$g(x,y) = \delta(x)$

$G(k_x,k_y) = \delta(k_y)$ !!!