Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2015
CT/Fourier Lecture 4



# Convolution/Modulation Theorem

$$
\begin{aligned}
F\{g(x)*h(x)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u)*h(x-u)du\right]e^{-j2\pi k_x x}dx \\
&= \int_{-\infty}^{\infty} g(u)\int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu \\
&= \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du \\
&= G(k_x)H(k_x)
\end{aligned}
$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x)*H(k_x)$$



# 2D Convolution/Multiplication

*Convolution*

$$F[g(x,y)**h(x,y)] = G(k_x,k_y)H(k_x,k_y)$$

*Multiplication*

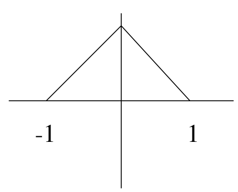
$$F[g(x,y)h(x,y)] = G(k_x,k_y)**H(k_x,k_y)$$



# Application of Convolution Thm.

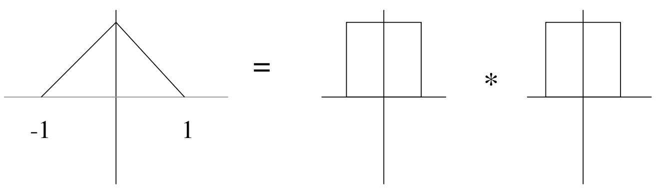
$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1}(1-|x|)e^{-j2\pi k_x x}dx = ??$$



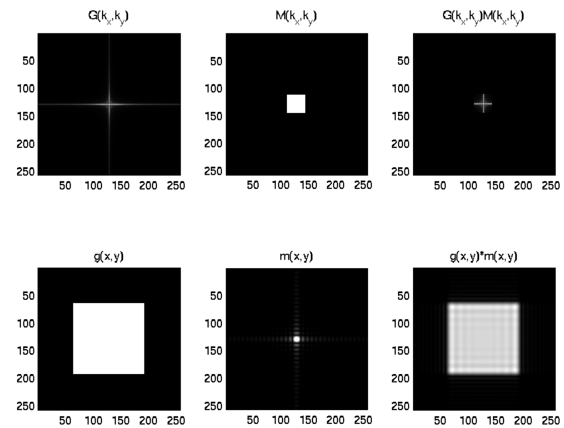
# Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$
$$F(\Lambda(x)) = \mathrm{sinc}^2(k_x)$$



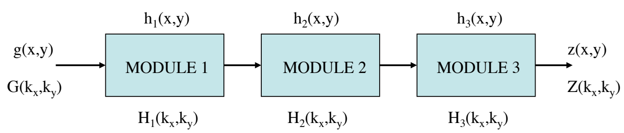



# Convolution Example






## Response of an Imaging System

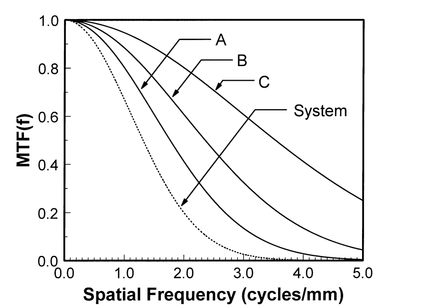



$$z(x,y)=g(x,y)**h_1(x,y)**h_2(x,y)**h_3(x,y)$$

$$Z(k_x,k_y)=G(k_x,k_y)\ H_1(k_x,k_y)\ H_2(k_x,k_y)\ H_3(k_x,k_y)$$



# System MTF = Product of MTFs of Components




Bushberg et al 2001

PollEv.com/be280a

Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots FWHM_N^2}$$

*Example*

$$FWHM_1 = 1mm$$

$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$



**D74.** The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ______ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

PollEv.com/be280a

# Modulation Examples

$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

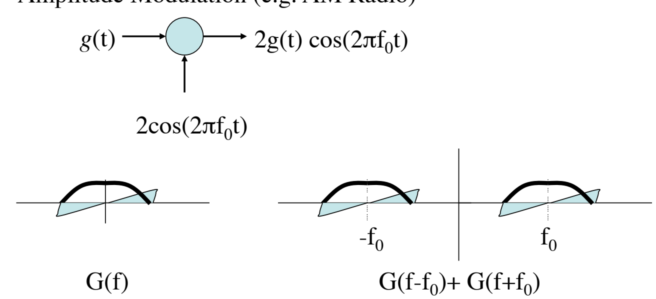
$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

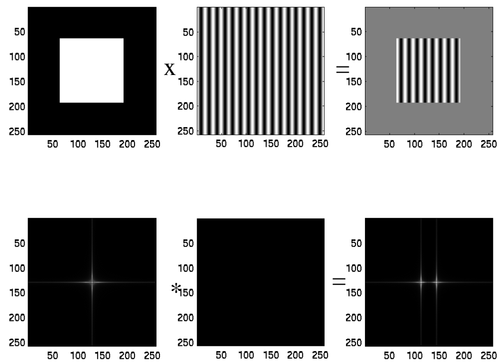


# Example

Amplitude Modulation (e.g. AM Radio)



# Modulation Example






# Shift Theorem

$$F\{g(x-a)\} = F\{g(x) * \delta(x-a)\}$$
$$= G(k_x)e^{-j2\pi a k_x}$$

$$F\{g(x-a,y-b)\} = F\{g(x,y) * \delta(x-a,y-b)\}$$
$$= G(k_x,k_y)e^{-j2\pi(k_x a + k_y b)}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.

Each frequency component is shifted by $a$. This corresponds to a relative phase shift of

$-2\pi a/(\text{spatial period}) = -2\pi a k_x$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by $a$ yields

$\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi a k_x)$



# Summary of Basic Properties

*Linearity*

$F[ag(x,y)+bh(x,y)] = aG(k_x,k_y)+bH(k_x,k_y)$

*Scaling*

$F[g(ax,by)] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$

*Duality*

$F\{G(x)\} = g(-k_x)$

*Shift*

$F[g(x-a,y-b)] = G(k_x,k_y)e^{-j2\pi(k_x a+k_y b)}$

*Convolution*

$F[g(x,y)**h(x,y)] = G(k_x,k_y)H(k_x,k_y)$

*Multiplication*

$F[g(x,y)h(x,y)] = G(k_x,k_y)**H(k_x,k_y)$

*Modulation*

$F[g(x,y)e^{j2\pi(xa+yb)}] = G(k_x-a,k_y-b)$

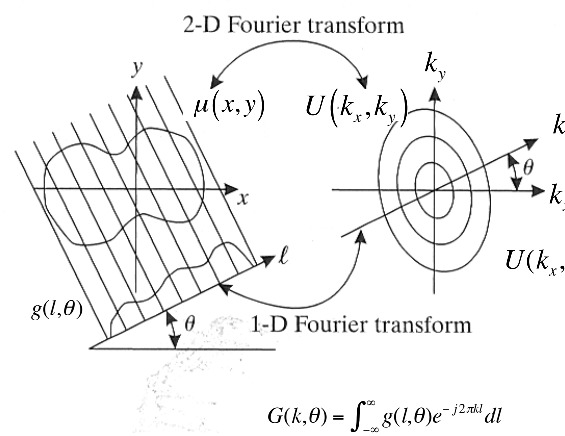


# Projection-Slice Theorem

$k_x = k\cos\theta$

$k_y = k\sin\theta$

$k = \sqrt{k_x^2 + k_y^2}$




$U(k_x,k_y) = G(k,\theta)$

$G(k,\theta) = \int_{-\infty}^{\infty} g(l,\theta)e^{-j2\pi kl}\,dl$



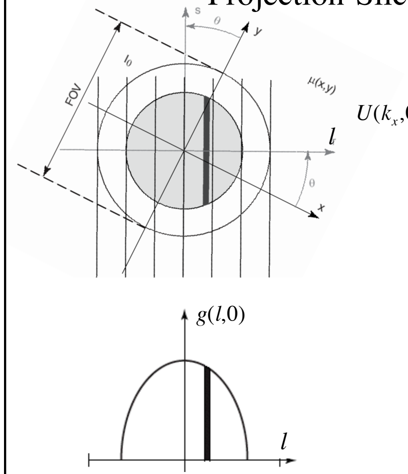
Modified from Prince&Links 2006

# Projection-Slice Theorem

$$G(k,\theta) = \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi kl} dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi kl}\, dx\ dy\ dl$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi k(x\cos\theta + y\sin\theta)} dx\ dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dx\ dy$$
$$= F_{2D}\left[\mu(x,y)\right]\Big|_{k_x = k\cos\theta, k_y = k\sin\theta}$$



# Projection-Slice Theorem



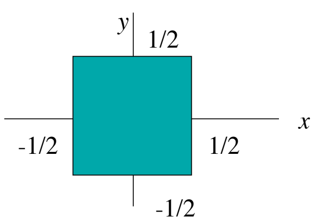

$$U(k_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(k_x x + k_y y)} dxdy$$
$$= \int_{-\infty}^{-\infty}\left[\int_{-\infty}^{-\infty} \mu(x,y) dy\right] e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{-\infty} g(x,0) e^{-j2\pi k_x x} dx$$
$$= \int_{-\infty}^{-\infty} g(l,0) e^{-j2\pi kl} dl$$

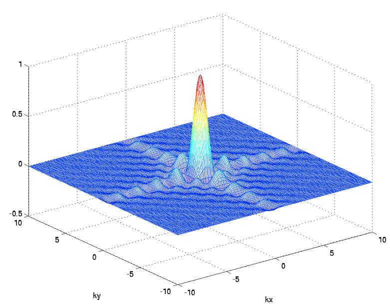


Suetens 2002

# Example (sinc/rect)

*Example*
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$






# Example (sinc/rect)

*Example*
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Projection at $\theta = 0$;
$g(l,0) = rect(l) \rightarrow F(g(l,0)) = \mathrm{sinc(k)}$
$k_x = k\cos\theta = k;\ \ k_y = k\sin\theta = 0$
$G(k_x,k_y) = \mathrm{sinc}(k)\mathrm{sinc}(0) = \mathrm{sinc}(k)$

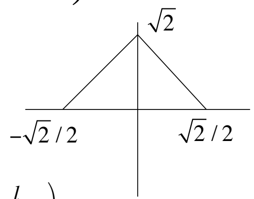
Projection at $\theta = 90$;
$g(l,90) = rect(l) \rightarrow F(g(l,90)) = \mathrm{sinc(k)}$
$k_x = k\cos\theta = 0;\ \ k_y = k\sin\theta = k$
$G(k_x,k_y) = \mathrm{sinc}(0)\mathrm{sinc}(k) = \mathrm{sinc}(k)$

# Example (sinc/rect)

*Example*

$g(x,y)=\Pi(x)\Pi(y)$

$G(k_x,k_y)=\mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$




Projection at $\theta = 45$;

$$g(l,45)=\sqrt{2}\Lambda\left(\frac{l}{\sqrt{2}/2}\right)=2rect\left(\frac{l}{\sqrt{2}/2}\right)*rect\left(\frac{l}{\sqrt{2}/2}\right)$$

$$\rightarrow F(g(l,45))=2\frac{\sqrt{2}}{2}\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)=\mathrm{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$

$$k_x=k\cos\theta=k\frac{\sqrt{2}}{2};\ \ k_y=k\sin\theta=k\frac{\sqrt{2}}{2}$$

$$G(k_x,k_y)=\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)\mathrm{sinc}\left(k\frac{\sqrt{2}}{2}\right)=\mathrm{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$



# Example (sinc/rect)

*Example*

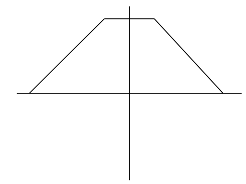
$g(x,y)=\Pi(x)\Pi(y)$

$G(k_x,k_y)=\mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Projection at $0<\theta<90$

$k_x=k\cos\theta;\ \ k_y=k\sin\theta$

$G(k_x,k_y)=\mathrm{sinc}(k\cos\theta)\mathrm{sinc}(k\sin\theta)$



$$g(l,\theta)=F^{-1}\left(\mathrm{sinc}(k\cos\theta)\mathrm{sinc}(k\sin\theta)\right)$$

$$=F^{-1}\left(\mathrm{sinc}(k\cos\theta)\right)*F^{-1}\left(\mathrm{sinc}(k\sin\theta)\right)$$

$$=\frac{1}{\cos\theta}rect\left(\frac{l}{\cos\theta}\right)*\frac{1}{\sin\theta}rect\left(\frac{l}{\sin\theta}\right)$$

$$=\frac{1}{\cos\theta\sin\theta}rect\left(\frac{l}{\cos\theta}\right)*rect\left(\frac{l}{\sin\theta}\right)$$



# In-class Exercise

$$\mu(x,y)=rect(x,y)\cos\left(2\pi(x+y)\right)$$

Sketch the object
Calculate and sketch the Fourier Transform of the Object



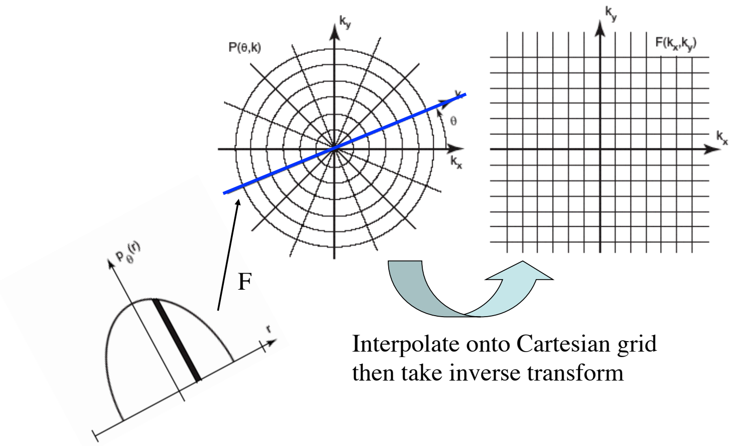
# In-class Exercise

$$\mu(x,y)=rect(x,y)\cos\left(2\pi(x+y)\right)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

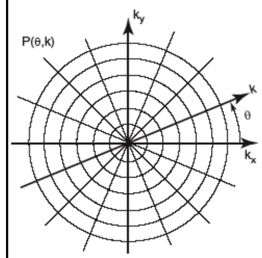
PollEv.com/be280a

# Fourier Reconstruction




 Suetens 2002

# Polar Version of Inverse FT




$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}kdkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta + y\sin\theta)}|k|dkd\theta$$

*Note* :

$g(l,\theta+\pi) = g(-l,\theta)$

*So*

$G(k,\theta+\pi) = G(-k,\theta)$

 Suetens 2002

# Filtered Backprojection

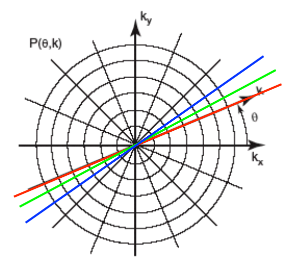
$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}|k|dkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dkd\theta$$
$$= \int_0^{\pi} g^*(l,\theta)d\theta \longleftarrow \text{Backproject a filtered projection}$$

$where\ l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dk$$
$$= g(l,\theta) * F^{-1}[|k|]$$
$$= g(l,\theta) * q(l)$$

 Suetens 2002

# Fourier Interpretation

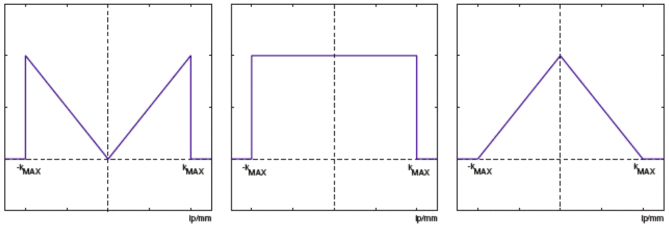



$$\text{Density} \approx \frac{\text{N}}{\text{circumference}} \approx \frac{\text{N}}{2\pi|\text{k}|}$$

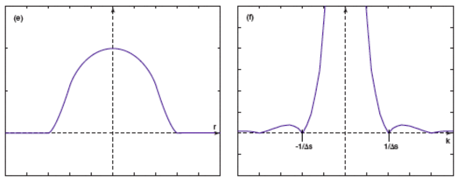
Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

# Ram-Lak Filter



$k_{max}=1/\Delta s$

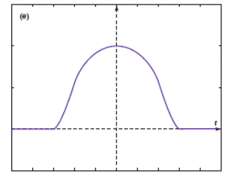




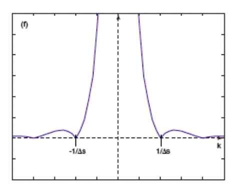
Suetens 2002

# Reconstruction Path

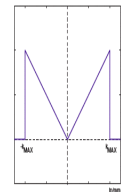
Projection



F ↓



x



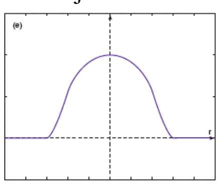
$F^{-1}$ → Filtered Projection → Back-Project



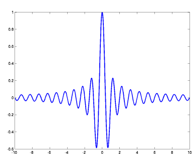
Suetens 2002

# Reconstruction Path

Projection



*



→ Filtered Projection

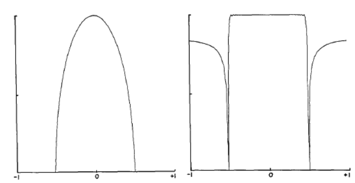
↓

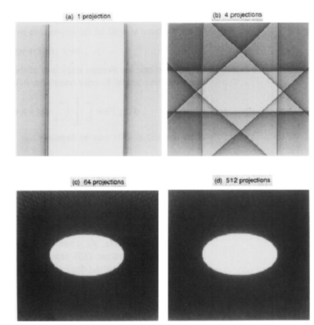
Back-Project



Suetens 2002

# Example



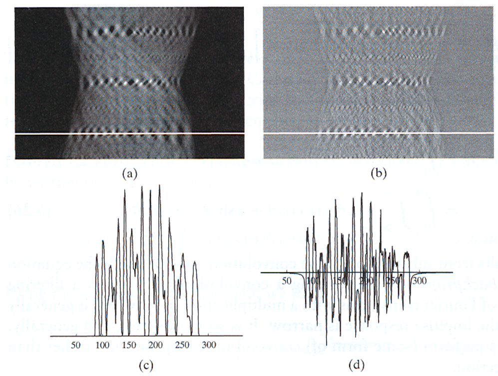




Kak and Slaney

# Example

**Figure 6.15**
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].




Prince and Links 2005



# Example

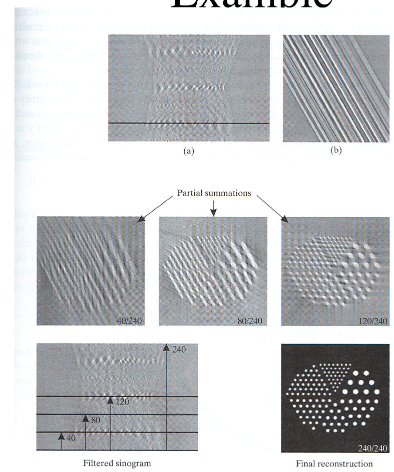
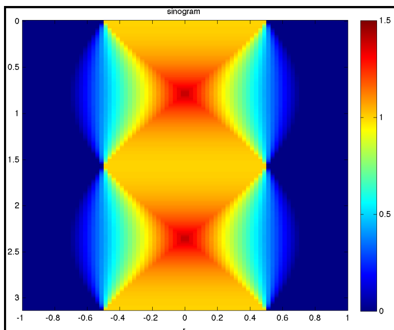


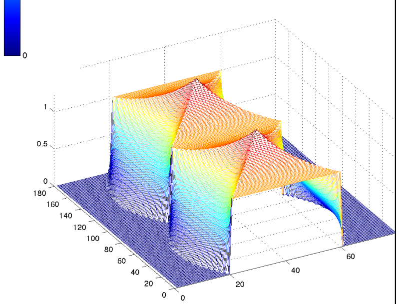

**Figure 6.16**
Backprojection step.

**Figure 6.17**
Summation step.

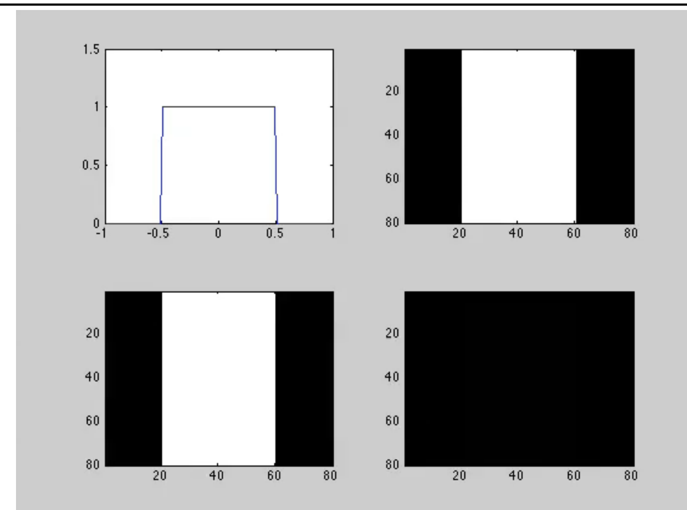
Prince and Links 2005

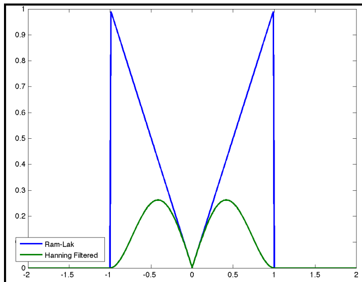




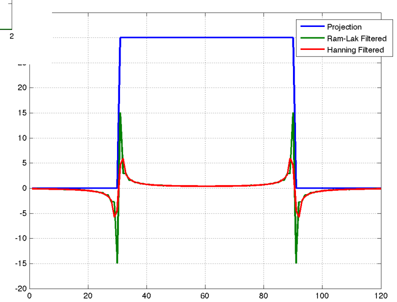




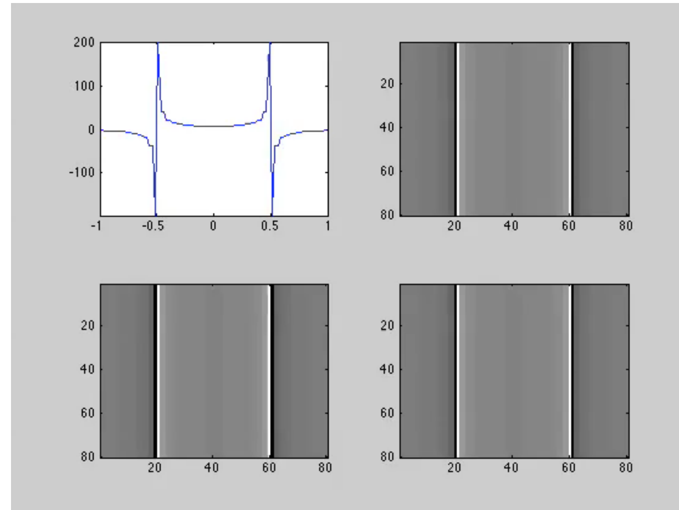



Filters

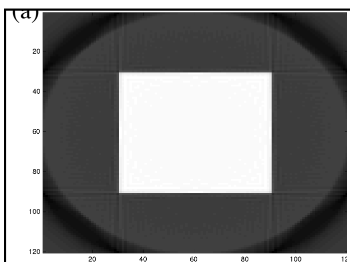
Filtered Projections

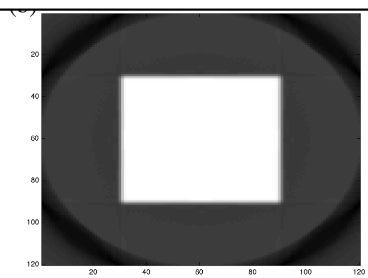


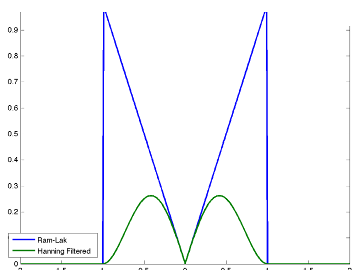











Which image used the Hanning filter?

PollEv.com/be280a