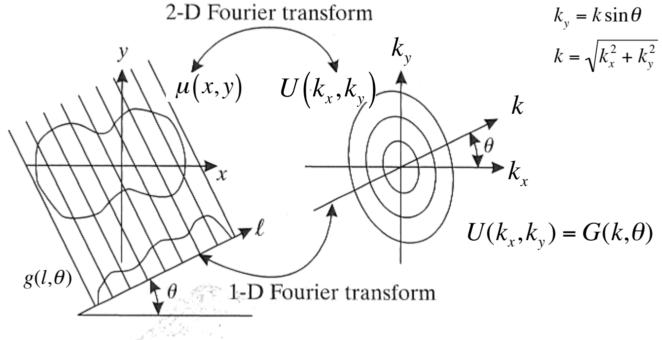
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2015
CT/Fourier Lecture 5

Projection-Slice Theorem

$k_x = k\cos\theta$
$k_y = k\sin\theta$
$k = \sqrt{k_x^2 + k_y^2}$



$G(k,\theta) = \int_{-\infty}^{\infty} g(l,\theta)e^{-j2\pi kl}dl$

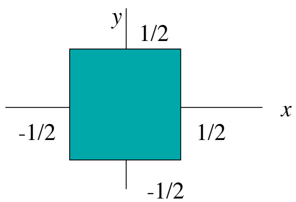
Modified from Prince&Links 2006

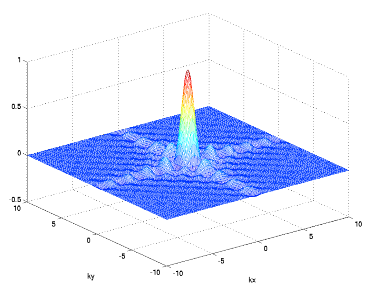
Example (sinc/rect)

Example

$\mu(x,y) = \Pi(x)\Pi(y)$

$U(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$





Example (sinc/rect)

Example

$\mu(x,y) = \Pi(x)\Pi(y)$

$U(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$

Projection at $\theta = 0$;

$g(l,0) = rect(l) \rightarrow F(g(l,0)) = \text{sinc(k)}$

$k_x = k\cos\theta = k;\quad k_y = k\sin\theta = 0$

$U(k_x,k_y)\big|_{k\cos\theta, k\sin\theta} = \text{sinc}(k)\text{sinc}(0) = \text{sinc}(k)$

Projection at $\theta = 90$;

$g(l,90) = rect(l) \rightarrow F(g(l,90)) = \text{sinc(k)}$

$k_x = k\cos\theta = 0;\quad k_y = k\sin\theta = k$

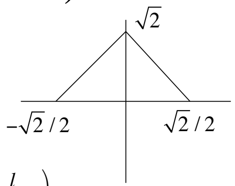
$U(k_x,k_y)\big|_{k\cos\theta, k\sin\theta} = \text{sinc}(0)\text{sinc}(k) = \text{sinc}(k)$

Example (sinc/rect)

Example

$\mu(x,y) = \Pi(x)\Pi(y)$

$U(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$



Projection at $\theta = 45$;

$$g(l,45) = \sqrt{2}\Lambda\left(\frac{l}{\sqrt{2}/2}\right) = 2rect\left(\frac{l}{\sqrt{2}/2}\right) * rect\left(\frac{l}{\sqrt{2}/2}\right)$$

$$\rightarrow F(g(l,45)) = 2\frac{\sqrt{2}}{2}\text{sinc}\left(k\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}\text{sinc}\left(k\frac{\sqrt{2}}{2}\right) = \text{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$

$$k_x = k\cos\theta = k\frac{\sqrt{2}}{2};\quad k_y = k\sin\theta = k\frac{\sqrt{2}}{2}$$

$$U(k_x,k_y)\big|_{k\cos\theta,k\sin\theta} = \text{sinc}\left(k\frac{\sqrt{2}}{2}\right)\text{sinc}\left(k\frac{\sqrt{2}}{2}\right) = \text{sinc}^2\left(k\frac{\sqrt{2}}{2}\right)$$

Example (sinc/rect)

Example

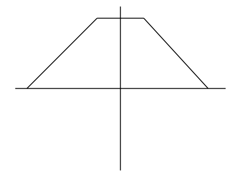
$\mu(x,y) = \Pi(x)\Pi(y)$

$U(k_x,k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$

Projection at $0 < \theta < 90$

$k_x = k\cos\theta;\quad k_y = k\sin\theta$

$$U(k_x,k_y)\big|_{k\cos\theta,k\sin\theta} = \text{sinc}(k\cos\theta)\text{sinc}(k\sin\theta)$$

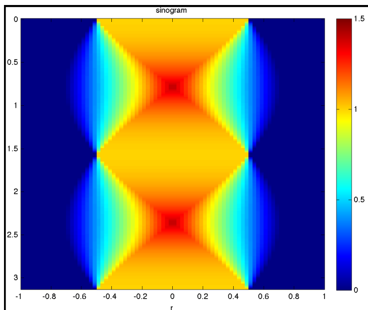


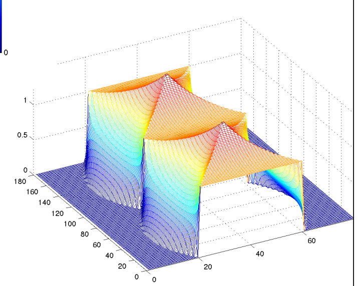
$$g(l,\theta) = F^{-1}\left(\text{sinc}(k\cos\theta)\text{sinc}(k\sin\theta)\right)$$

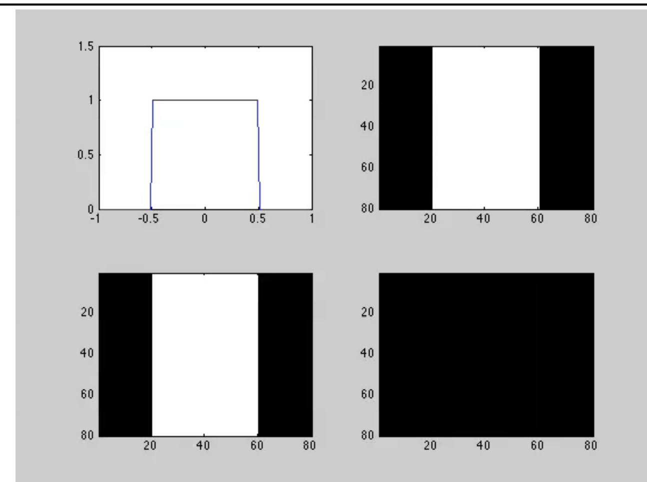
$$= F^{-1}\left(\text{sinc}(k\cos\theta)\right) * F^{-1}\left(\text{sinc}(k\sin\theta)\right)$$

$$= \frac{1}{\cos\theta}rect\left(\frac{l}{\cos\theta}\right) * \frac{1}{\sin\theta}rect\left(\frac{l}{\sin\theta}\right)$$

$$= \frac{1}{\cos\theta\sin\theta}rect\left(\frac{l}{\cos\theta}\right) * rect\left(\frac{l}{\sin\theta}\right)$$







In-class Exercise

$$\mu(x,y) = rect(x,y)\cos\left(2\pi(x+y)\right)$$

Sketch this object.
What are the projections at theta = 0 and 90 degrees?
For what angle is the projection maximized?

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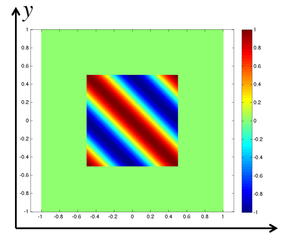
$\mu(x,y) = rect(x,y)\cos\left(2\pi(x+y)\right)$



Peaks occur along the lines $y = n - x$ where n is an integer.

Valleys occur along the lines

$$y = \frac{2n+1}{2} - x$$

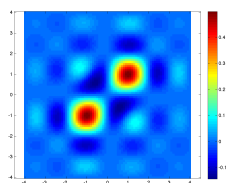
Projection at 0 degrees

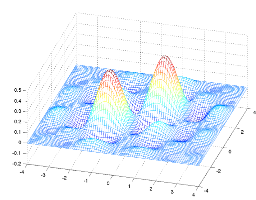
$$g(l) = \int_{-1/2}^{1/2} \cos\left(2\pi(l+y)\right)dy$$
$$= \frac{1}{2\pi}\sin\left(2\pi(l+y)\right)\Big|_{-1/2}^{1/2}$$
$$= \frac{1}{2\pi}\left(\sin\left(2\pi\left(l+\frac{1}{2}\right)\right) - \sin\left(2\pi\left(l-\frac{1}{2}\right)\right)\right)$$
$$= 0$$

Projection at 90 degrees

$$g(l) = \int_{-1/2}^{1/2} \cos\left(2\pi(x+l)\right)dx$$
$$= 0$$

$$U(k_x,k_y) = \frac{1}{2}\left(\mathrm{sinc}(k_x-1,k_y-1) + \mathrm{sinc}(k_x+1,k_y+1)\right)$$





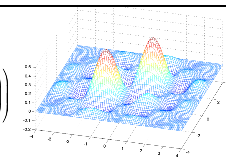
Fourier transform of projection at 0 degrees:

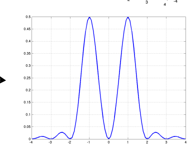
$$F\left[g(l)\right] = U(k_x,k_y)\big|_{k,0}$$
$$= \frac{1}{2}\left(\mathrm{sinc}(k-1,0-1) + \mathrm{sinc}(k+1,0+1)\right)$$
$$= \frac{1}{2}\left(\mathrm{sinc}(k-1)\mathrm{sinc}(-1) + \mathrm{sinc}(k+1)\mathrm{sinc}(1)\right)$$
$$= 0$$

---

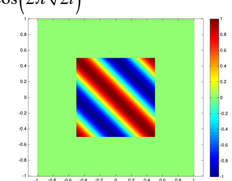
at 45 degrees

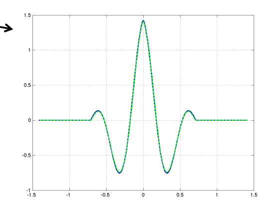
$$F\left[g(l)\right] = U(k_x,k_y)\big|_{\frac{\sqrt{2}}{2}k,\frac{\sqrt{2}}{2}k}$$
$$= \frac{1}{2}\left(\mathrm{sinc}\left(\frac{\sqrt{2}}{2}k-1,\frac{\sqrt{2}}{2}k-1\right) + \mathrm{sinc}\left(\frac{\sqrt{2}}{2}k+1,\frac{\sqrt{2}}{2}k+1\right)\right)$$
$$= \frac{1}{2}\left(\mathrm{sinc}^2\left(\frac{\sqrt{2}}{2}\left(k-\sqrt{2}\right)\right) + \mathrm{sinc}^2\left(\frac{\sqrt{2}}{2}\left(k+\sqrt{2}\right)\right)\right)$$
$$= \mathrm{sinc}^2\left(\frac{\sqrt{2}}{2}k\right) * \frac{1}{2}\left(\delta\left(k-\sqrt{2}\right) + \delta\left(k+\sqrt{2}\right)\right)$$



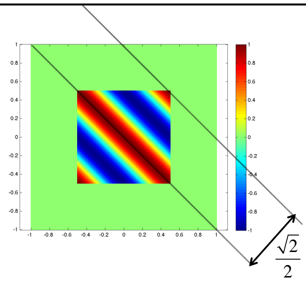


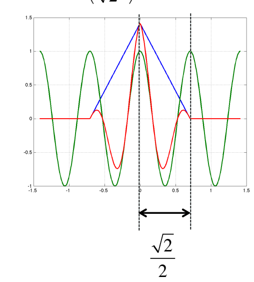
$$g(l) = \left(\frac{2}{\sqrt{2}}\mathrm{rect}\left(\frac{2}{\sqrt{2}}l\right) * \frac{2}{\sqrt{2}}\mathrm{rect}\left(\frac{2}{\sqrt{2}}l\right)\right)\cdot\cos\left(2\pi\sqrt{2}l\right)$$
$$= \sqrt{2}\Lambda\left(\frac{2}{\sqrt{2}}l\right)\cos\left(2\pi\sqrt{2}l\right)$$



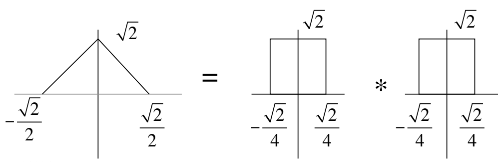


$$g(l) = \sqrt{2}\Lambda\left(\frac{2}{\sqrt{2}}l\right)\cos\left(2\pi\sqrt{2}l\right)$$



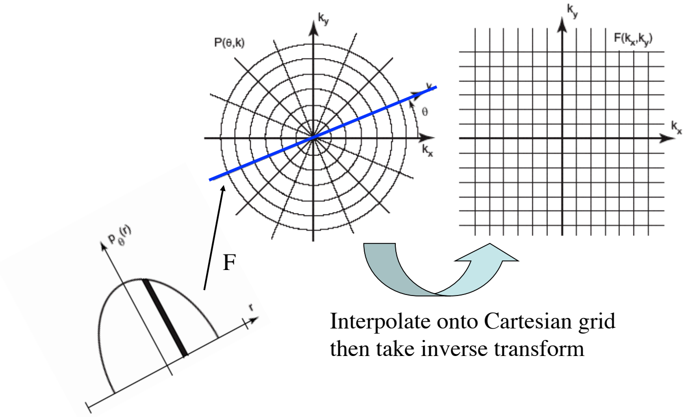


$$\left(\frac{2}{\sqrt{2}}\mathrm{rect}\left(\frac{2}{\sqrt{2}}l\right) * \frac{2}{\sqrt{2}}\mathrm{rect}\left(\frac{2}{\sqrt{2}}l\right)\right) = \sqrt{2}\Lambda\left(\frac{2}{\sqrt{2}}l\right)$$



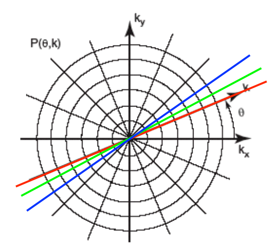

```
dx = 0.01;xax = -1:dx:1;
[x,y] = meshgrid(xax,xax);
z = (abs(x)<=0.5 &  abs(y)<=0.5);
z0 = z.*cos(2*pi*(x+y));

figure(1);  % The object
imagesc(xax,xax,z0);colorbar;
axis equal; set(gca,'YDir','normal');
 figure(2);% Image of Fourier Transform
kax = -4:.1:4;
[kx,ky] = meshgrid(kax,kax);
z1 = 0.5*(sinc(kx-1).*sinc(ky-1) + sinc(kx+1).*sinc(ky+1));
imagesc(kax,kax,z1);colorbar;axis equal; set(gca,'YDir','normal');
 figure(3);%  Mesh plot of Fourier Transform
mesh(kax,kax,z1);
axis([-4 4 -4 4 -0.2 0.5]);
view(16,48);
 figure(4);% Fourier Transform along 45 degree line.
plot(diag(kx),diag(z1),'LineWidth',2);grid;
 figure(5);% approximate the projection at 45 degrees.
% and compare to exact formulation
[nr,nc] = size(z0) ;
ind = toeplitz(nr:-1:1,[1:nc]+nr-1) ;
z0t = flipud(z0);;
% approximation
proj = accumarray(ind(:),z0t(:))*sqrt(2)*dx;
pax = (dx*sqrt(2)/2)*(-(length(proj)-1)/2:(length(proj)-1)/2);
% exact formulation.
proj2 = sqrt(2).*cos(2*pi*sqrt(2)*pax).*tripuls(pax,sqrt(2));
plot(pax,proj,'b',pax,proj2,'g--','LineWidth',2);grid;
 figure(6); % plot consituent parts of the projection at 45 degrees.
plot(pax,sqrt(2)*tripuls(pax,sqrt(2)),pax,cos(2*pi*sqrt(2)*pax),pax,proj2,'LineWidth',2);
grid;
```


Fourier Reconstruction



Suetens 2002

Fourier Interpretation

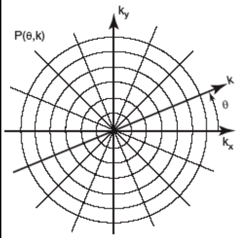


$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

Polar Version of Inverse FT



$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}kdkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta + y\sin\theta)}|k|dkd\theta$$

Note:

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

Suetens 2002

Filtered Backprojection

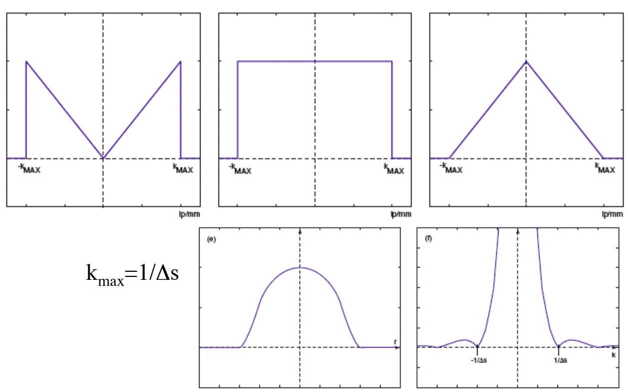
$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}|k|dkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dkd\theta$$
$$= \int_0^{\pi} g^*(l,\theta)d\theta \longleftarrow \text{Backproject a filtered projection}$$

where $l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty}|k|G(k,\theta)e^{j2\pi kl}dk$$
$$= g(l,\theta) * F^{-1}\left[|k|\right]$$
$$= g(l,\theta) * q(l)$$

Suetens 2002

Ram-Lak Filter

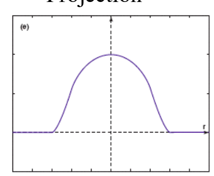


$k_{max}=1/\Delta s$

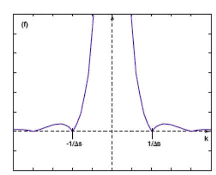
Suetens 2002

Reconstruction Path

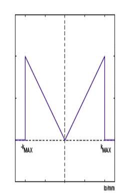
Projection



F ↓



x

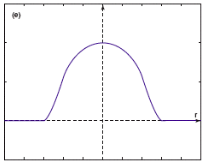


F^{-1} → Filtered Projection → Back-Project

Suetens 2002

Reconstruction Path

Projection



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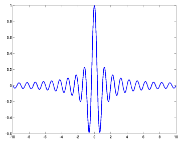

⟶ Filtered Projection

↓

Back-Project

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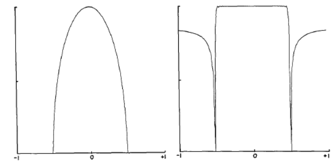
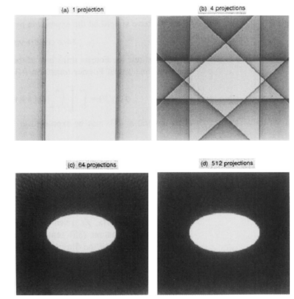
Suetens 2002

Example





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Kak and Slaney

Example

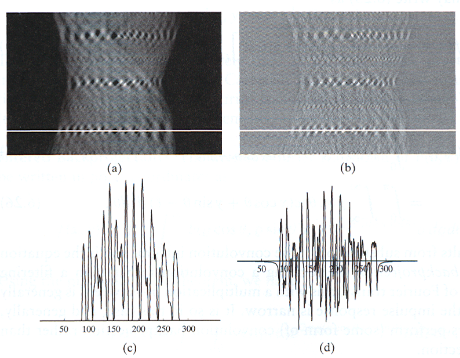

Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].

Prince and Links 2005

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Example

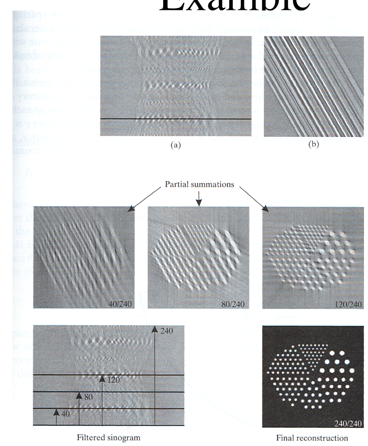

Figure 6.16
Backprojection step.

Figure 6.17
Summation step.

Prince and Links 2005

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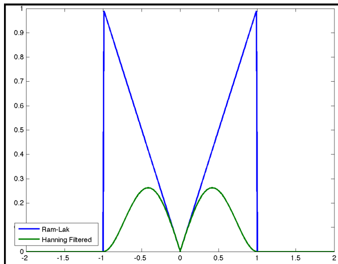
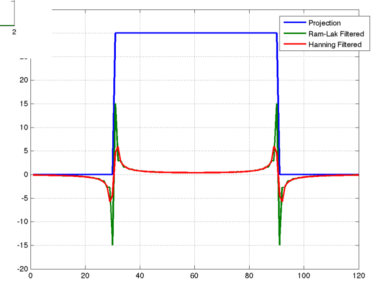


Filters

Filtered Projections



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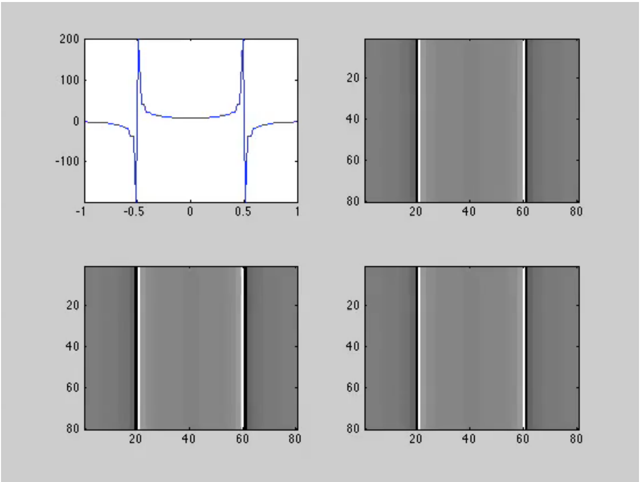

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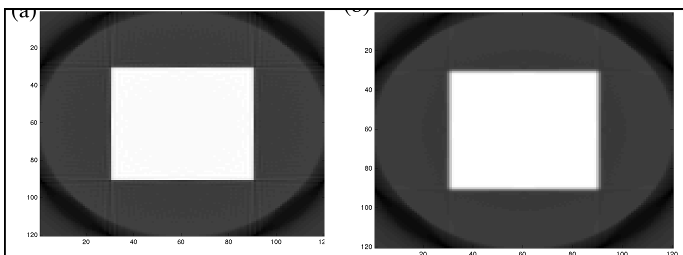

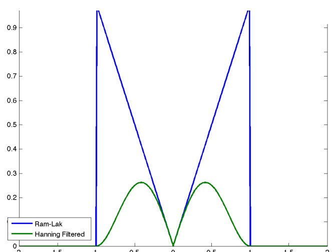

Which image used the Hanning filter?

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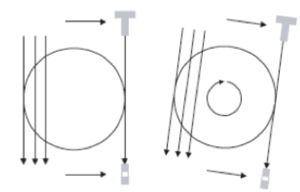
CT Sampling Requirements

What should the size of the detectors be?

How many detectors (or lines) do we need?

How many views do we need?



(a)

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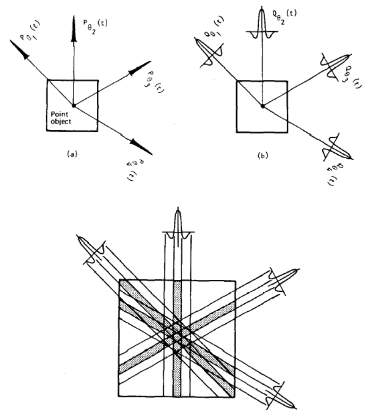
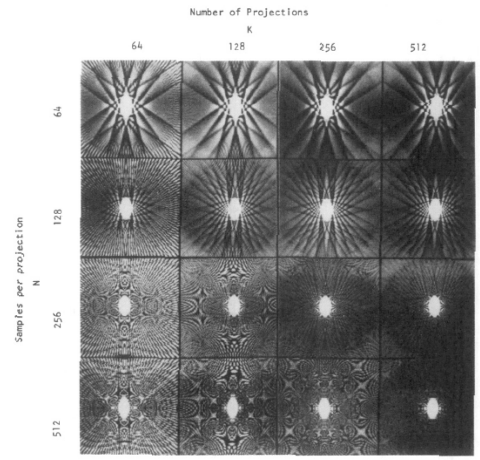
Suetens 2002

View Aliasing



Kak and Slaney



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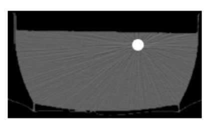
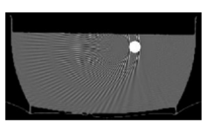
Artifacts

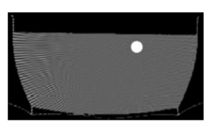
Object



Effect of Noise



Aliasing due to insufficient number of detectors



Aliasing due to insufficient number of views

TT Liu, BE280A, UCSD Fall 2015

Suetens 2002

Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

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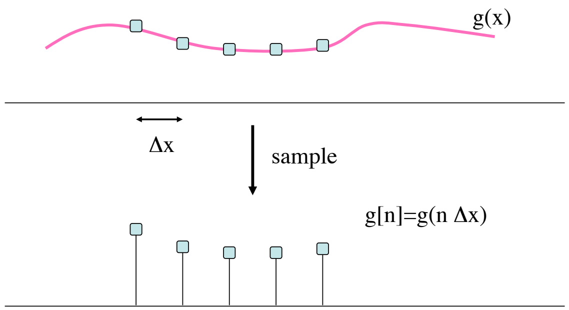
The Process of Sampling



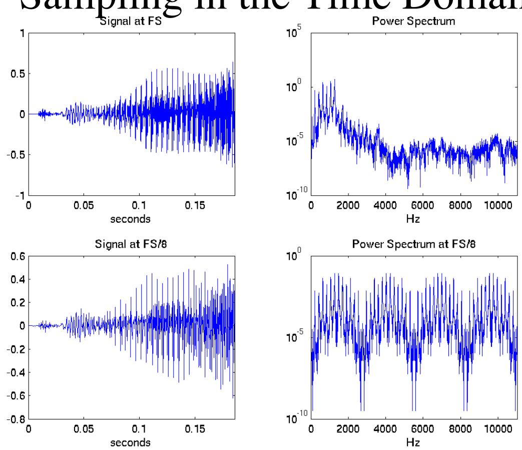
Questions

How finely do we need to sample?

What happens if we don't sample finely enough?

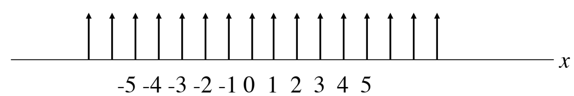
Can we reconstruct the original signal or image from its samples?

Sampling in the Time Domain



Comb Function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



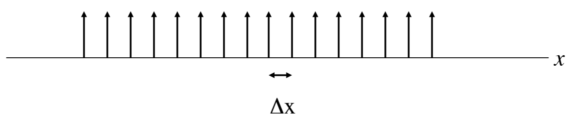
Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$

$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x)\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

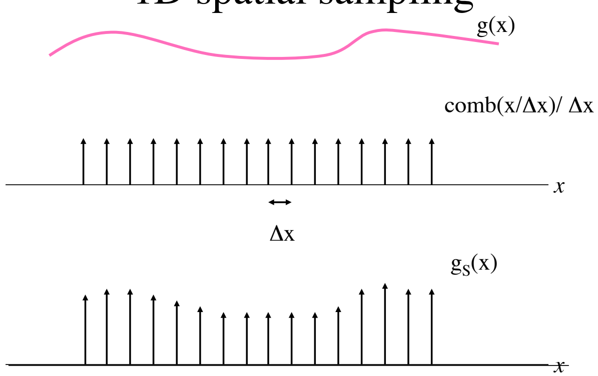
$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty} \delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

1D spatial sampling



Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$

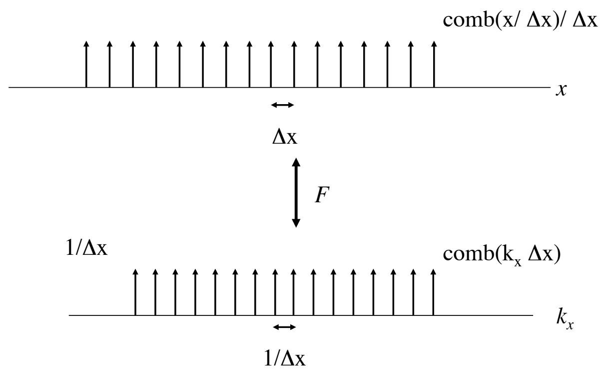
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)\right] = \frac{1}{\Delta x}\Delta x\, comb(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{n}{\Delta x}\right)$$

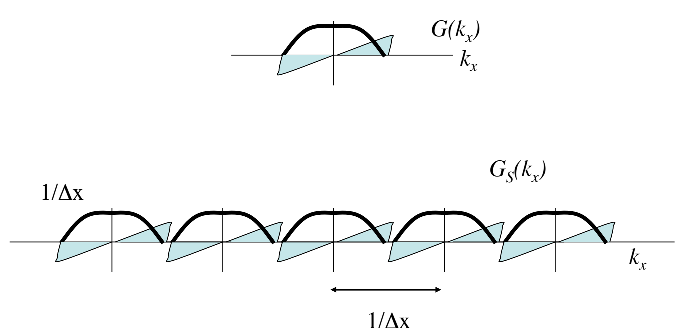
Fourier Transform of comb(x/ Δx)



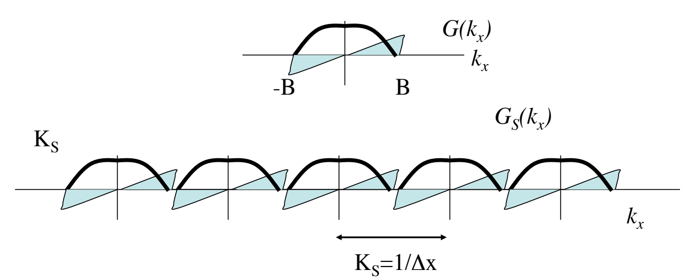
Fourier Transform of $g_S(x)$

$$
\begin{aligned}
F\left[g_S(x)\right] &= F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G\left(k_x - \frac{n}{\Delta x}\right)
\end{aligned}
$$

Fourier Transform of $g_S(x)$



Nyquist Condition



To avoid overlap, we require that 1/Δx>2B or
K_S > 2B where K_S=1/ Δx is the sampling frequency

Example

Assume that the highest spatial frequency in an object is B = 2 cm^{-1}.
Thus, smallest spatial period is 0.5 cm.

Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

Example

Assume that the Nyquist sampling periods of $f(x)$ and $g(x)$
are Δf and Δg, respectively. Determine the Nyquist sampling periods for
$a) f(x - x_0)$
$b) f(x) + g(x)$
$c) f(x) * f(x)$

PollEv.com/be280a

from Prince and Links 2006