Signal Equation

Demodulate the signal to obtain

\[ s(t) = e^{i\omega_0 t} s(t) \]

\[ = \int \int m(x, y) \exp \left( -2\pi j \int G(\tau) \cdot \tau d\tau \right) dx \, dy \]

\[ = \int \int m(x, y) \exp \left( -2\pi j \int [G_x(\tau) x + G_y(\tau) y] \, d\tau \right) dx \, dy \]

\[ = \int \int m(x, y) \exp \left( -2\pi j \int \int (k_x(\tau) x + k_y(\tau) y) \, d\tau \right) dx \, dy \]

Where

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0 G_x(\tau) \, d\tau \]

\[ k_y(t) = \frac{\gamma}{2\pi} \int_0 G_y(\tau) \, d\tau \]

K-space

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M[k_x(t), k_y(t)] = F[m(x, y)] \]

evaluated at the spatial frequencies:

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0 G_x(\tau) \, d\tau \]

\[ k_y(t) = \frac{\gamma}{2\pi} \int_0 G_y(\tau) \, d\tau \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.
Units

Spatial frequencies ($k_x$, $k_y$) have units of 1/distance. Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau$$

$$= [Hz/Gauss][Gauss/cm][sec] = [1/cm]$$

Example

$G_x(t) = 1$ Gauss/cm

$$k_x(t_2) = \frac{\gamma}{2\pi} \int_0^{t_2} G_x(\tau) d\tau$$

$$= \frac{4257 Hz}{G \cdot 1G/cm \cdot 0.235 \times 10^{-3} s}$$

$$= 1 \text{ cm}^{-1}$$

K-space trajectory

$$k_x(t_2) = \frac{\gamma}{2\pi} \int_0^{t_2} G_x(\tau) d\tau$$

$$= [Hz/Gauss][Gauss/cm][sec] = [1/cm]$$
K-space trajectory

Spin-Warp
Spin-Warp Pulse Sequence

K-space trajectories

Very Expensive Bagpipes
i.e.
Prompting fMRI Subjects with Gradient Waveforms
Sampling in k-space

Aliasing

Intuitive view of Aliasing

$\Delta B_z(x) = G_x x$

Slower

Faster

$\frac{1}{FOV}$

$\frac{2}{FOV}$
Instead of sampling the signal, we sample its Fourier Transform.

\[
G_S(kx) = G(kx) \delta(n - \frac{kx - n\Delta k}{\Delta k})
\]

\[
\chi = \frac{1}{\Delta k} \text{comb}(kx/\Delta k)
\]

To avoid overlap, \(1/\Delta k > \text{FOV}\), or equivalently, \(\Delta k < 1/\text{FOV}\).
### Aliasing

Aliasing occurs when \(\frac{1}{\Delta k_x} < \text{FOV}\)

### 2D Comb Function

\[
\text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m) \delta(y - n)
\]

\[
= \text{comb}(x) \text{comb}(y)
\]

### Scaled 2D Comb Function

\[
\text{comb}(x/\Delta x, y/\Delta y) = \text{comb}(x/\Delta x) \text{comb}(y/\Delta y)
\]

\[
= \Delta x \Delta y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x) \delta(y - n\Delta y)
\]
2D k-space sampling

\[ G_s(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb} \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \]

\[ = G(k_x, k_y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y) \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

Nyquist Conditions

Windowing

Windowing the data in Fourier space

\[ G_w(k_x, k_y) = G(k_x, k_y)W(k_x, k_y) \]

Results in convolution of the object with the inverse transform of the window

\[ g_w(x, y) = g(x, y) \ast w(x, y) \]
Resolution

$W(k_x, k_y) = \text{rect} \left( \frac{k_x}{W_{k_x}} \right) \cdot \text{rect} \left( \frac{k_y}{W_{k_y}} \right)$

$w(x, y) = F^{-1} \left[ \text{rect} \left( \frac{k_x}{W_{k_x}} \right) \cdot \text{rect} \left( \frac{k_y}{W_{k_y}} \right) \right]$

$= W_{k_x} W_{k_y} \sin(c(W_{k_x} x)) \sin(c(W_{k_y} y))$

$g_w(x, y) = g(x, y) \ast W_{k_x} W_{k_y} \sin(c(W_{k_x} x)) \sin(c(W_{k_y} y))$

Windowing Example

Effective Width

$W_E = \frac{1}{w(0)} \int_{-\infty}^{\infty} w(x) \, dx$

Example

$w(x) = \frac{1}{W_{k_x}} \int_{-\infty}^{\infty} \sin(c(W_{k_x} x)) \, dx$

$= \left[ F[\sin(c(W_{k_x} x))] \right]_{-\infty}^{\infty}$

$= \frac{1}{W_{k_x}} \text{rect} \left( \frac{k}{W_{k_x}} \right)_{-\infty}^{\infty}$

$= \frac{1}{W_{k_x}}$

Resolution and spatial frequency

With a window of width $W_E$, the highest spatial frequency is $W_E/2$.

This corresponds to a spatial period of $2/W_E$. 
Windowing Example

\[ g(x, y) = [\delta(x) + \delta(x-1)] \delta(y) \]

\[ g_k(x, y) = [\delta(x) + \delta(x-1)] \delta(x) \ast \ast W_k \ast \ast \frac{\text{sinc}(W_k x)}{\text{sinc}(W_k y)} = W_k \frac{\text{sinc}(W_k x)}{\text{sinc}(W_k y)} \]

Sampling and Windowing

Sampling and windowing the data in Fourier space

\[ G_{sw}(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb} \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \text{rect} \left( \frac{k_x}{W_k x}, \frac{k_y}{W_k y} \right) \]

Results in replication and convolution in object space.

\[ g_{sw}(x, y) = W_k \ast \ast \frac{\text{sinc}(W_k x)}{\text{sinc}(W_k y)} \ast \ast \text{comb}(\Delta k_x, \Delta k_y) \ast \ast \text{sinc}(W_k x) \text{sinc}(W_k y) \]
**Sampling in $k_x$**

- RF Signal
  - $\cos \omega_0 t$
  - $\sin \omega_0 t$

  **ADC**

  **Low pass Filter**

  **Note:** In practice, there are number of ways of implementing this processing.

  $M = I + jQ$

**Sampling in $k_y$**

- RF
- $G_x(t)$
- $G_y(t)$

  $\tau_y$

  $\Delta k_y = \frac{\gamma}{2\pi} G_{\chi} \tau_y$

  $FOV_y = \frac{1}{\Delta k_y}$

**Sampling in $k_x$**

- $G_x(t)$

  $\Delta k_x = \frac{\gamma}{2\pi} G_{\chi} \Delta t$

  $FOV_x = \frac{1}{\Delta k_x}$
Resolution

\[ \delta_x = \frac{1}{W_x} = \frac{1}{2k_{x,\text{max}}} = \frac{1}{2\pi G_x \tau_x} \]

\[ \delta_y = \frac{1}{W_y} = \frac{1}{2k_{y,\text{max}}} = \frac{1}{2\pi G_y \tau_y} \]

\[ G_x(t) \]

\[ G_y(t) \]

Example

Goal:

\[ \text{FOV}_x = \text{FOV}_y = 25.6 \text{ cm} \]

\[ \delta_x = \delta_y = 0.1 \text{ cm} \]

Readout Gradient:

\[ G_x = \frac{1}{\text{FOV}_x \gamma 2\pi} \]

\[ \Delta t = 32 \text{ usec} \]

\[ G_x = \frac{1}{\text{FOV}_x \gamma 2\pi} \]

\[ \tau_x = \frac{1}{\delta_x \gamma 2\pi} \]

\[ \tau_y = \frac{1}{\delta_y \gamma 2\pi} \]

where

\[ N_{\text{max}} = \frac{\text{FOV}_x}{\Delta} = 256 \]

Example

Phase-Encode Gradient:

\[ G_y = \frac{1}{\text{FOV}_y \gamma 2\pi} \]

\[ \tau_y = \frac{1}{\delta_y \gamma 2\pi} \]

\[ \gamma = 1 \times 10^4 \text{ Tesla} \]

\[ G_{y1} \]

\[ G_{y2} \]

ADC

\[ \Delta t \]
Example

Phase - Encode Gradient:
\[ \delta = \frac{120 \pi \tau_y}{2 \pi} \]
\[ G_y = \frac{1}{\delta} \frac{4257 G}{4.096 \times 10^{-4} s} \]
\[ = 0.2868 \frac{G}{cm} \]
\[ = \frac{N_y}{\delta y} G_y \]

where
\[ N_y = \frac{FOV}{\delta y} = 256 \]

Example

Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with \( \Delta t = 10 \) µsec. The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly.

Assume \( \gamma (2\pi) = 4000 Hz / G \)

Sampling

In practice, an even number (typically power of 2) sample is usually taken in each direction to take advantage of the Fast Fourier Transform (FFT) for reconstruction.

Gibbs Artifact

256x256 image

256x128 image
Apodization

\[ h(k_x) = \frac{1}{2}(1 + \cos(2\pi k_x)) \]

\[ \text{Hanning Window} \]

\[ s(x) = 0.5\text{sinc}(x) + 0.25\text{sinc}(x-1) + 0.25\text{sinc}(x+1) \]

Aliasing and Bandwidth

\[ B = \gamma G_s \text{FOV}_x \]

Lowpass filter in the readout direction to prevent aliasing.

Temporal filtering in the readout direction limits the readout FOV. So there should never be aliasing in the readout direction.