K-space

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M(k_x(t), k_y(t)) = F[m(x, y)](k_x(t), k_y(t)) \]

evaluated at the spatial frequencies:

\[ k_x(t) = \gamma \frac{2\pi}{\tau} \int_0^\tau G_x(\tau) d\tau \]
\[ k_y(t) = \gamma \frac{2\pi}{\tau} \int_0^\tau G_y(\tau) d\tau \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

Spin-Warp Pulse Sequence

<table>
<thead>
<tr>
<th>RF</th>
<th>( G_x(t) ) (Readout, or Frequency) Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( G_y(t) ) (Phase Encode Gradient)</td>
</tr>
<tr>
<td></td>
<td>( G_z(t) ) (Phase Encodes)</td>
</tr>
</tbody>
</table>
**Sampling and Windowing**

\[ \Delta k_x < 1/(\text{FOV}_x) \]
\[ \Delta k_y < 1/(\text{FOV}_y) \]

\[ W_{k_x} > 1/\delta_x \]
\[ W_{k_y} > 1/\delta_y \]

---

**Example**

**Goal:**

\[ \text{FOV}_x = \text{FOV}_y = 25.6 \text{ cm} \]
\[ \delta_x = \delta_y = 0.1 \text{ cm} \]

Readout Gradient:

\[ G_x = \frac{1}{\Delta t} \frac{\gamma}{2\pi} \text{FOV} \]

Pick \( \Delta t = 32 \mu\text{sec} \):

\[ G_x = \frac{1}{\text{FOV}} \frac{\gamma}{2\pi} \Delta t \]

\[ = \frac{1}{(25.6 \times 10^{-2} \text{ m} \times 42.57 \times 10^7 \text{ Hz/G} \times 32 \times 10^{-6})} \]

\[ = 2.8675 \times 10^{-3} \text{ T/cm} \]

\[ = 0.28675 \text{ G/cm} \]

1 Gauss = \( 1 \times 10^{-4} \) Tesla

---

**Example**

Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with \( \Delta t = 10 \mu\text{sec} \). The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly.

Assume \( \gamma/(2\pi) = 4000 \text{Hz/G} \)
Gibbs Artifact

256x256 image

256x128 image

Images from http://www.mritutor.org/mritutor/gibbs.htm

Apodization

\[ h(k_x) = \frac{1}{2} \left( 1 + \cos(2\pi k_x) \right) \]

\( \text{rect}(k_x) \)

Hanning Window

\[ 0.5\text{sinc}(x) + 0.25\text{sinc}(x-1) + 0.25\text{sinc}(x+1) \]

Aliasing and Bandwidth

Lowerpass filter in the readout direction to prevent aliasing.

Readout

Faster

Slower

\( \text{FOV}_x \)

GE Medical Systems 2003
Temporal filtering in the readout direction limits the readout FOV. So there should never be aliasing in the readout direction.

From Buxton 2002
Free Induction Decay (FID)

An excitation pulse rotates the magnetization vector away from its equilibrium state (purely longitudinal). The resulting vector has both longitudinal $M_z$ and transverse $M_{xy}$ components.

Due to thermal interactions, the magnetization will return to its equilibrium state with characteristic time constants.

$T_1$, spin-lattice time constant, return to equilibrium of $M_z$

$T_2$, spin-spin time constant, return to equilibrium of $M_{xy}$

Relaxation

Longitudinal Relaxation

\[
\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}
\]

After a 90 degree pulse

\[
M_z(t) = M_0(1 - e^{-t/T_1})
\]

Due to exchange of energy between nuclei and the lattice (thermal vibrations). Process continues until thermal equilibrium as determined by Boltzmann statistics is obtained.

The energy $\Delta E$ required for transitions between down to up spins, increases with field strength, so that $T_1$ increases with $B$.

T1 Values

Gray Matter, muscle
White matter
Kidney, liver
Fat
Transverse Relaxation

\[ \frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2} \]

Each spin’s local field is affected by the z-component of the field due to other spins. Thus, the Larmor frequency of each spin will be slightly different. This leads to a dephasing of the transverse magnetization, which is characterized by an exponential decay.

\[ T_2 \] is largely independent of field. \( T_2 \) is short for low frequency fluctuations, such as those associated with slowly tumbling macromolecules.

T2 Relaxation

After a 90 degree excitation

\[ M_{xy}(t) = M_0 e^{-t/T_2} \]

T2 Values

<table>
<thead>
<tr>
<th>Tissue</th>
<th>( T_2 ) (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray matter</td>
<td>100</td>
</tr>
<tr>
<td>white matter</td>
<td>92</td>
</tr>
<tr>
<td>muscle</td>
<td>47</td>
</tr>
<tr>
<td>fat</td>
<td>85</td>
</tr>
<tr>
<td>kidney</td>
<td>58</td>
</tr>
<tr>
<td>liver</td>
<td>43</td>
</tr>
<tr>
<td>CSF</td>
<td>4000</td>
</tr>
</tbody>
</table>

Solids exhibit very short \( T_2 \) relaxation times because there are many low frequency interactions between the immobile spins. On the other hand, liquids show relatively long \( T_2 \) values, because the spins are highly mobile and net fields average out.
**Example**

Questions: How can one achieve T2 weighting? What are the relative $T_2$'s of the various tissues?

**Bloch Equation**

\[
\frac{dM}{dt} = M \times \gamma B - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_0)k}{T_1}
\]

$M$, $i$, $j$, $k$ are unit vectors in the $x,y,z$ directions.

**Precession**

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & -B_0 & 0 \\
-B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define $M = M_x + jM_y$.

\[
dM/dt = d/dt(M_x + jM_y) = -j\gamma B_0 M
\]

Solution is a time-varying phasor

\[
M(t) = M(0)e^{-j\omega_0 t} = M(0)e^{-j\gamma B_0 t}
\]

**Question:** which way does this rotate with time?

**Matrix Form with $B = B_0$**

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
-1/T_2 & \gamma B_0 & 0 \\
-\gamma B_0 & 1/T_2 & 0 \\
0 & 0 & -1/T_1
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
M_0/T_1
\end{bmatrix}
\]
**Z-component solution**

\[ M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1} \]

- **Saturation Recovery**
  - If \( M_z(0) = 0 \) then \( M_z(t) = M_0(1 - e^{-t/T_1}) \)

- **Inversion Recovery**
  - If \( M_z(0) = -M_0 \) then \( M_z(t) = M_0(1 - 2e^{-t/T_1}) \)

**Summary**

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

**Transverse Component**

\[ M = M_x + jM_y \]

\[ \frac{dM}{dt} = d/dt(M_x + iM_y) \]

\[ = -j(\omega_0 + 1/T_2)M \]

\[ M(t) = M(0)e^{-j\omega_0 t / e^{-1/T_2}} \]

**Summary**

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

**Fact:** Can show that \( T_2 < T_1 \) in order for \( |M(t)| \leq M_0 \)

Physically, the mechanisms that give rise to \( T_1 \) relaxation also contribute to transverse \( T_2 \) relaxation.
Static Inhomogeneities

In the ideal situation, the static magnetic field is totally uniform and the reconstructed object is determined solely by the applied gradient fields. In reality, the magnet is not perfect and will not be totally uniform. Part of this can be addressed by additional coils called “shim” coils, and the process of making the field more uniform is called “shimming.” In the old days this was done manually, but modern magnets can do this automatically.

In addition to magnet imperfections, most biological samples are inhomogeneous and this will lead to inhomogeneity in the field. This is because, each tissue has different magnetic properties and will distort the field.

Field Inhomogeneities

The spatial nonuniformity in the field can be modeled by adding an additional term to our signal equation.

\[ s(t) = \int M(\mathbf{r},t) dV \]

\[ = \int \int \int M(x,y,z,0)e^{-i \omega_0 t}e^{-i \mathbf{G} \mathbf{r} \cdot \tau} \exp(-j \int G(\mathbf{r}) \cdot \mathbf{r} d\mathbf{r}) dxdydz \]

The effect of this nonuniformity is to cause the spins to dephase with time and thus for the signal to decrease more rapidly. To first order this can be modeled as an additional decay term of the form

\[ s(t) = \int \int \int M(x,y,z,0)e^{-i \omega_0 t}e^{-i \mathbf{G} \mathbf{r} \cdot \tau} \exp(-j \int G(\mathbf{r}) \cdot \mathbf{r} d\mathbf{r}) dxdydz \exp(-t/T_2) \]
The overall decay has the form:

$$\exp\left(-t/T_2^*\right)$$

where

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

Due to random motions of spins. Not reversible. Due to static inhomogeneities. Reversible with a spin-echo sequence.

Gradient echo sequences exhibit $T_2^*$ decay.

Gradient echo has $\exp(-TE/T_2^*)$ weighting.

Spin Echo

Discovered by Erwin Hahn in 1950.

The spin-echo can refocus the dephasing of spins due to static inhomogeneities. However, there will still be $T_2$ dephasing due to random motion of spins.

There is nothing that nuclear spins will not do for you, as long as you treat them as human beings. Erwin Hahn

Source: http://mrsrl.stanford.edu/~brian/mri-movies/
Spin Echo

Phase at time $\tau$
$$\phi(\tau) = \int_0^\tau -\omega_E(\vec{r}) \, dt = -\omega_E(\vec{r}) \tau$$

Phase after 180 pulse
$$\phi(\tau + 180^\circ) = \omega_E(\vec{r}) \tau$$

Phase at time $2\tau$
$$\phi(2\tau) = -\omega_E(\vec{r}) \tau + \omega_E(\vec{r}) \tau = 0$$

Image Contrast

Different tissues exhibit different relaxation rates, $T_1$, $T_2$, and $T_2^*$. In addition different tissues can have different densities of protons. By adjusting the pulse sequence, we can create contrast between the tissues. The most basic way of creating contrast is adjusting the two sequence parameters: TE (echo time) and TR (repetition time).
Saturation Recovery Sequence

Gradient Echo
\[ I(x, y) = \rho(x, y) \left[ 1 - e^{-TR/T_1(x,y)} \right] e^{-TE/T_2^*(x,y)} \]

Spin Echo
\[ I(x, y) = \rho(x, y) \left[ 1 - e^{-TR/T_1(x,y)} \right] e^{-TE/T_2(x,y)} \]

T1-Weighted Scans
Make TE very short compared to either \( T_2 \) or \( T_2^* \). The resultant image has both proton and T1 weighting.

\[ I(x, y) \approx \rho(x, y) \left[ 1 - e^{-TR/T_1(x,y)} \right] \]

T2-Weighted Scans
Make TR very long compared to \( T_1 \) and use a spin-echo pulse sequence. The resultant image has both proton and T2 weighting.

\[ I(x, y) \approx \rho(x, y) e^{-TE/T_2} \]
T2-Weighted Scans

\[ I(x, y) \approx \rho(x, y) \]

Proton Density Weighted Scans

Make TR very long compared to \( T_1 \) and use a very short TE. The resultant image is proton density weighted.

Example

<table>
<thead>
<tr>
<th>Tissue</th>
<th>Proton Density</th>
<th>( T_1 ) (ms)</th>
<th>( T_2 ) (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Csf</td>
<td>1.0</td>
<td>400</td>
<td>2000</td>
</tr>
<tr>
<td>Gray</td>
<td>0.85</td>
<td>1250</td>
<td>120</td>
</tr>
<tr>
<td>White</td>
<td>0.7</td>
<td>850</td>
<td>90</td>
</tr>
</tbody>
</table>

Hanson 2009

(a) Four images, all obtained with a common TR=5 seconds and TE=90, 50, 20, 15 ms (shown in reading order).

(b) Six images obtained with a common TE=15 ms and TR=500, 1000, 2000, 3000, 4000, 5000 ms (shown in reading order).

1. Which has the longest \( T_1 \)?
2. Which has the shortest \( T_1 \)?
3. Which has the longest \( T_2 \)?
4. Which has the shortest \( T_2 \)?
5. Which might be pure water?
6. Which has the most firm jello?
a) Which is the most T1 weighted?
b) Which is the most T2 weighted?
c) Which is the most PD weighted?

FLASH sequence

\[ I(x,y) = \rho(x,y) \left[ 1 - e^{-\frac{TR}{T_1(x,y)}} \right] \sin \theta \]

\[ = \rho(x,y) \left[ 1 - e^{-\frac{TR}{T_1(x,y)}} \right] \cos \theta \]

\[ \times \exp \left( -\frac{TE}{T_2(x,y)} \right) \]

Signal intensity is maximized at the Ernst Angle

\[ \theta_E = \cos^{-1} \left( \exp \left( -\frac{TR}{T_1} \right) \right) \]

FLASH equation assumes no coherence from shot to shot. In practice this is achieved with RF spoiling.

Inversion Recovery

\[ I(x,y) = \rho(x,y) \left[ 1 - 2e^{-\frac{TI}{T_1(x,y)}} + e^{-\frac{TR}{T_1(x,y)}} \right] e^{-\frac{TE}{T_2(x,y)}} \]

Intensity is zero when inversion time is

\[ TI = -\frac{T_1 \ln \left( 1 + \exp \left( -\frac{TR}{T_1} \right) \right)}{2} \]
Inversion Recovery

Biglands et al. Journal of Cardiovascular Magnetic Resonance