Bioengineering 280B
Principles of Biomedical Imaging

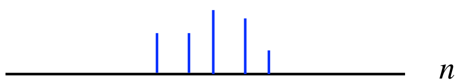
Spring Quarter 2005
Lecture 1
Linear Systems

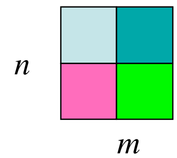
Topics

1. Linearity
2. Impulse Response and Delta functions
3. Superposition Integral
4. Shift Invariance
5. 1D and 2D convolution
6. Signal Representations

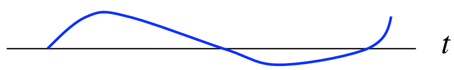
Signals and Images

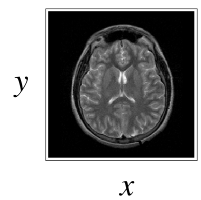
Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.



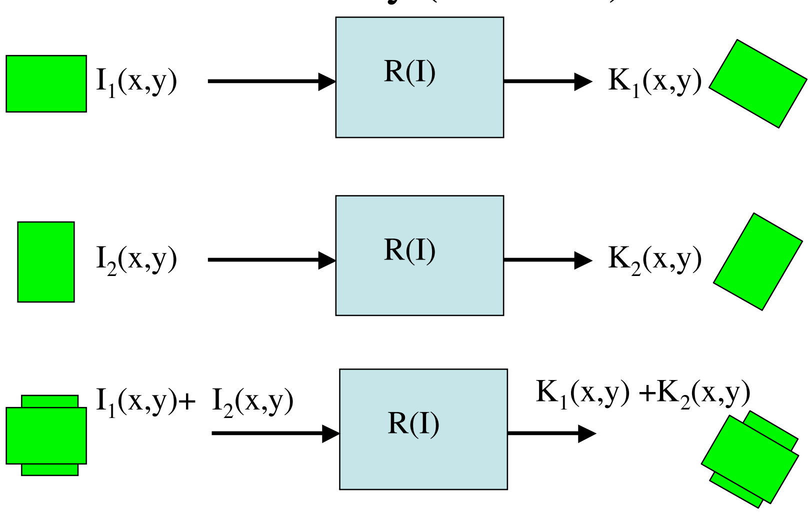


Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.

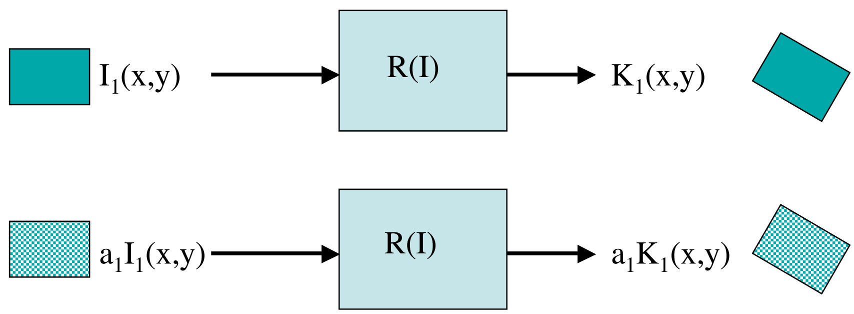




Linearity (Addition)



Linearity (Scaling)



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

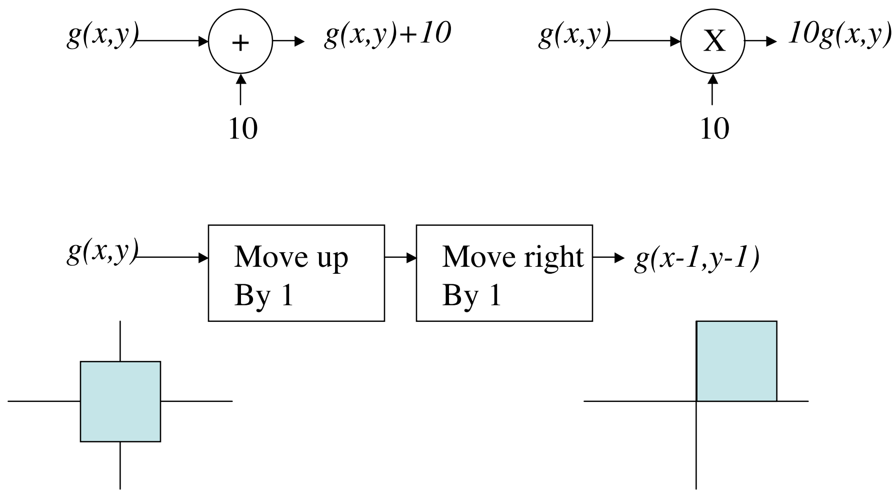
the response to the weighted sum of inputs is the weighted sum of outputs:

$$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$$

Thomas Liu, BE280B, UCSD, Spring 2005

Example

Are these linear systems?



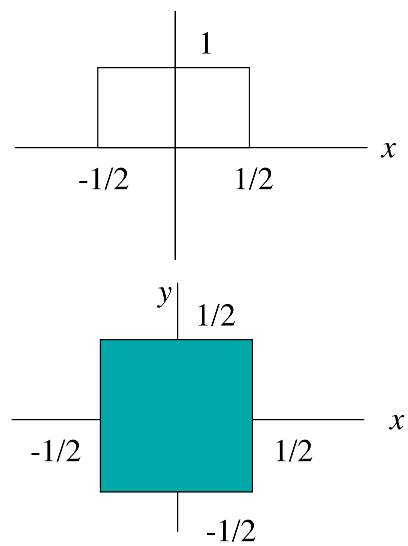
Thomas Liu, BE280B, UCSD, Spring 2005

Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \leq 1/2 \end{cases}$$

Also called rect(x)

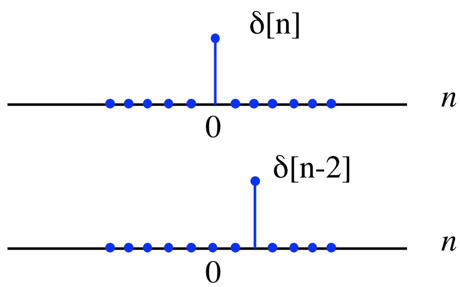
$$\Pi(x,y) = \Pi(x)\Pi(y)$$



Thomas Liu, BE280B, UCSD, Spring 2005

Kronecker Delta Function

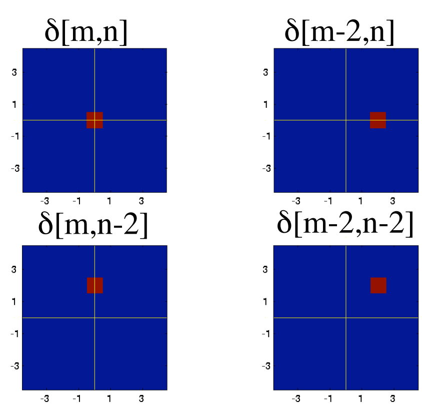
$$\delta[n] \quad = \begin{cases} 1 & \text{for } n = 0 \\ 0 & otherwise \end{cases}$$



Thomas Liu, BE280B, UCSD, Spring 2005

Kronecker Delta Function

$$\delta[m,n] \quad = \begin{cases} 1 & \text{for } m = 0, n = 0 \\ 0 & otherwise \end{cases}$$

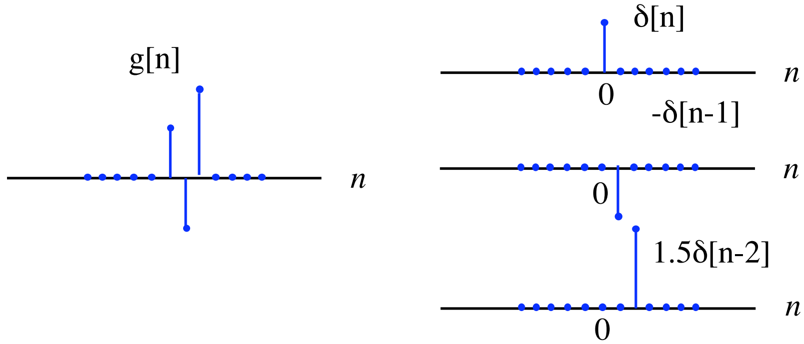


Thomas Liu, BE280B, UCSD, Spring 2005

Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



Thomas Liu, BE280B, UCSD, Spring 2005

Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

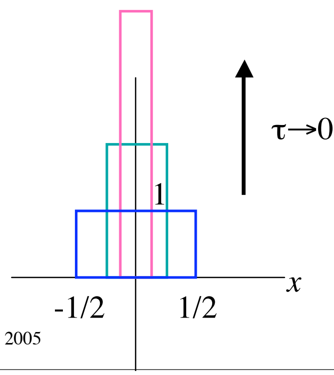
$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

Thomas Liu, BE280B, UCSD, Spring 2005

1D Dirac Delta Function

$\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



Thomas Liu, BE280B, UCSD, Spring 2005

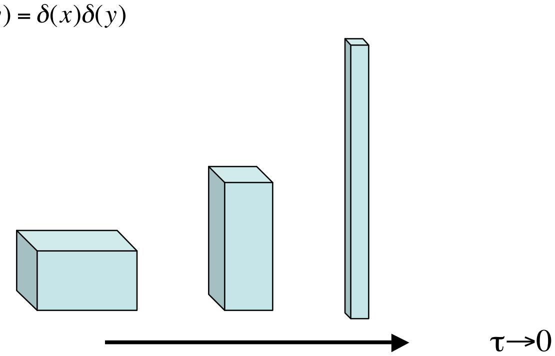
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \lim_{\tau \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-2}\Pi(x/\tau, y/\tau)dxdy.$

Useful fact : $\delta(x,y) = \delta(x)\delta(y)$

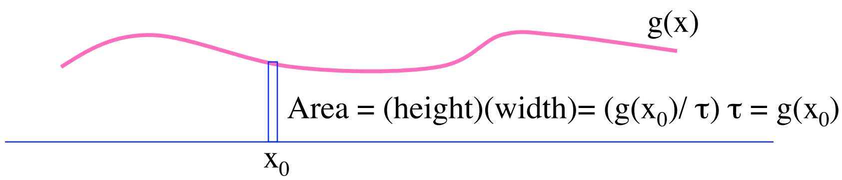


Thomas Liu, BE280B, UCSD, Spring 2005

Generalized Functions

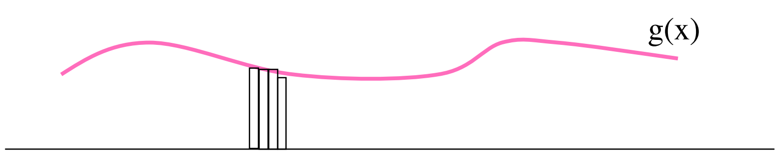
Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty} \delta(x - x_0) g(x) dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case $\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1} \Pi(x/\tau) g(x) dx = g(x_0)$



Representation of 1D Function

From the sifting property, we can write a 1D function as $g(x) = \int_{-\infty}^{\infty} g(\xi) \delta(x - \xi) d\xi$. To gain intuition, consider the approximation $g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x) \frac{1}{\Delta x} \Pi\left(\frac{x - n\Delta x}{\Delta x}\right) \Delta x.$



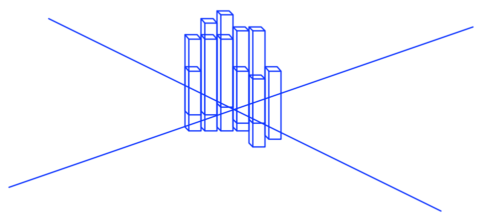
Representation of 2D Function

Similarly, we can write a 2D function as

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$$

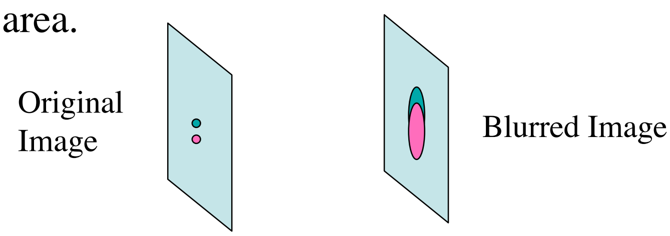
To gain intuition, consider the approximation

$$g(x,y)\approx\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x,m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$



Impulse Response

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



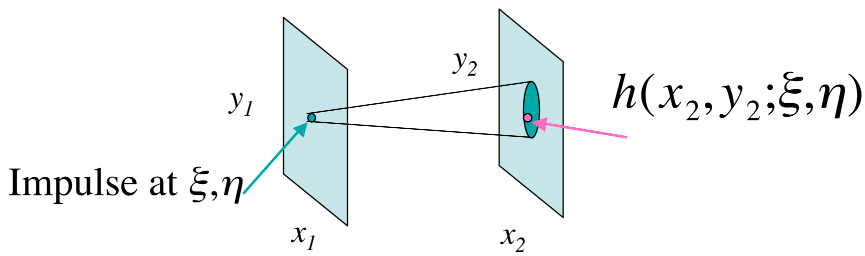
Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).

Impulse Response

The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L\left[\delta(x_1 - \xi)\right]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L\left[\delta(x_1 - \xi, y_1 - \eta)\right]$ 2D Impulse Response



Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

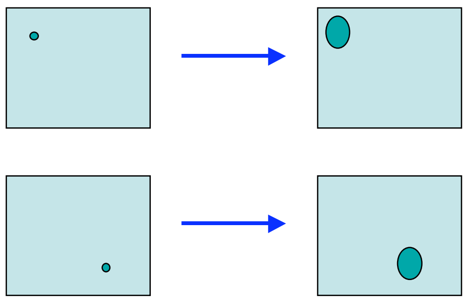
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)\,d\xi d\eta.$

The response is given by

$$\begin{aligned}
I(x_2,y_2) &= L\left[g_1(x_1,y_1)\right] \\
&= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)\,d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L\left[\delta(x_1-\xi,y_1-\eta)\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)\,d\xi d\eta
\end{aligned}$$

Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2-\xi,y_2-\eta)$



Thomas Liu, BE280B, UCSD, Spring 2005

2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

$$
\begin{aligned}
I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\
&= g(x_2,y_2)**h(x_2,y_2)
\end{aligned}
$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2)= g(x_2, y_2)*h(x_2, y_2)$.

Thomas Liu, BE280B, UCSD, Spring 2005

1D Convolution

For completeness, here is the 1D version.

$$
\begin{aligned}
I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\
&= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\
&= g(x)*h(x)
\end{aligned}
$$

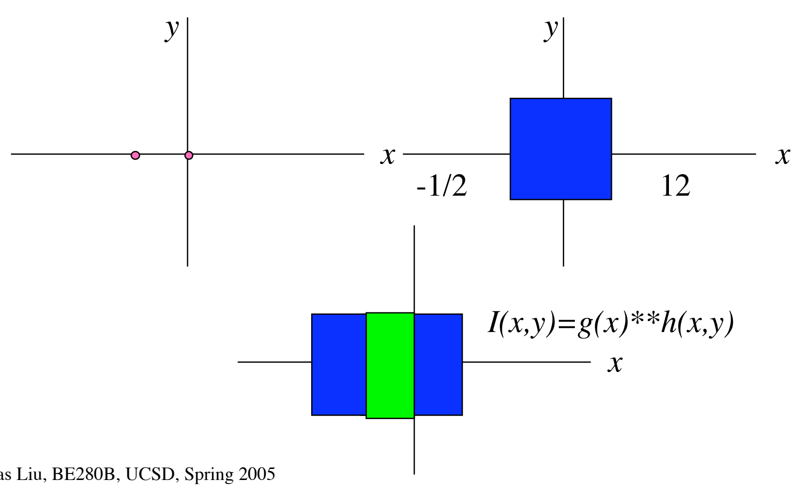
Useful fact:

$$
\begin{aligned}
g(x)*\delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\
&= g(x-\Delta)
\end{aligned}
$$

Thomas Liu, BE280B, UCSD, Spring 2005

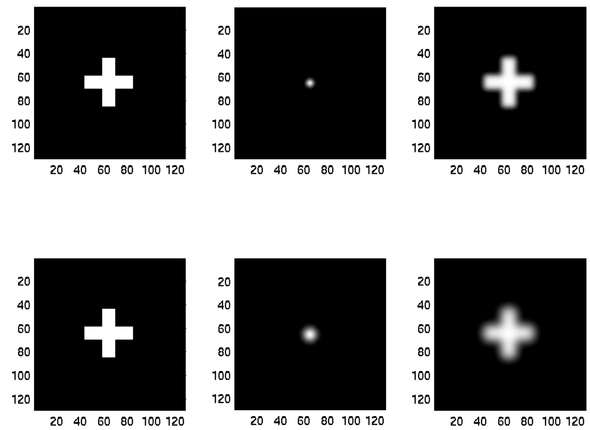
2D Convolution Example

g(x)= δ*(x+1/2,y) +* δ*(x,y)* *h(x)=rect(x,y)*



Thomas Liu, BE280B, UCSD, Spring 2005

2D Convolution Example

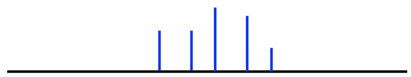


Summary

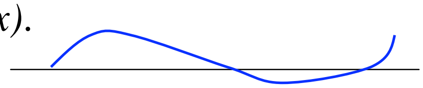
1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.

What is a signal?

Discrete-time/space signal: continuous valued function with a discrete time/space index, denoted as *s[n]*.

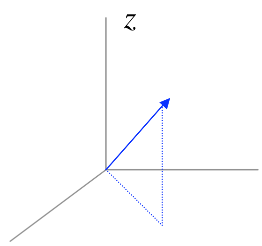


Continuous-time/space signal: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)*.



Signal Representation

It's easiest to start with discrete-time signals, which can be represented as vectors of either finite or infinite dimension. We'll start with finite dimensional vectors since they are easier to think about. Consider a finite-time signal with just 3 points. This can represented as a vector in $\Re^3$ for real-valued signals or C^3 for complex-valued signals.

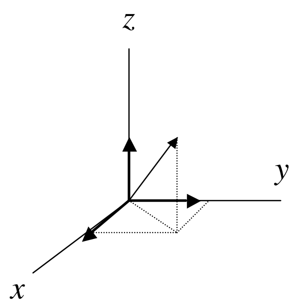


In signal notation: $s[n] = 1,1,1$

In vector notation: $\mathbf{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Basis Vectors

The numbers that we use to represent a signal depend on the choice of *basis vectors*, or more generally, *basis functions*.



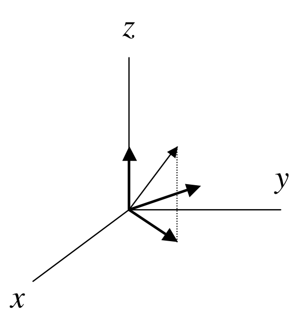
$$\mathbf{s} = \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}1 + \begin{bmatrix}0\\1\\0\end{bmatrix}1 + \begin{bmatrix}0\\0\\1\end{bmatrix}1$$

Here the unit vectors are used as the basis vectors. Note these are just Kronecker Delta functions!

Thomas Liu, BE280B, UCSD, Spring 2005

Basis Vectors

Any 3 vectors that *span* 3-dimensional space may be used as basis vectors. Recall from linear algebra, that these 3 vectors must be linearly independent. In other words, any one basis vector cannot be expressed as a linear sum of the other basis vectors. For any basis set, the signal coefficients are simply the weights of the basis vectors.



$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\sqrt{2} + \begin{bmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{bmatrix}0 + \begin{bmatrix}0\\0\\1\end{bmatrix}1$$

With this set of basis vectors, the coefficients of the signal are $s[n] = \sqrt{2}, 0, 1$

Thomas Liu, BE280B, UCSD, Spring 2005

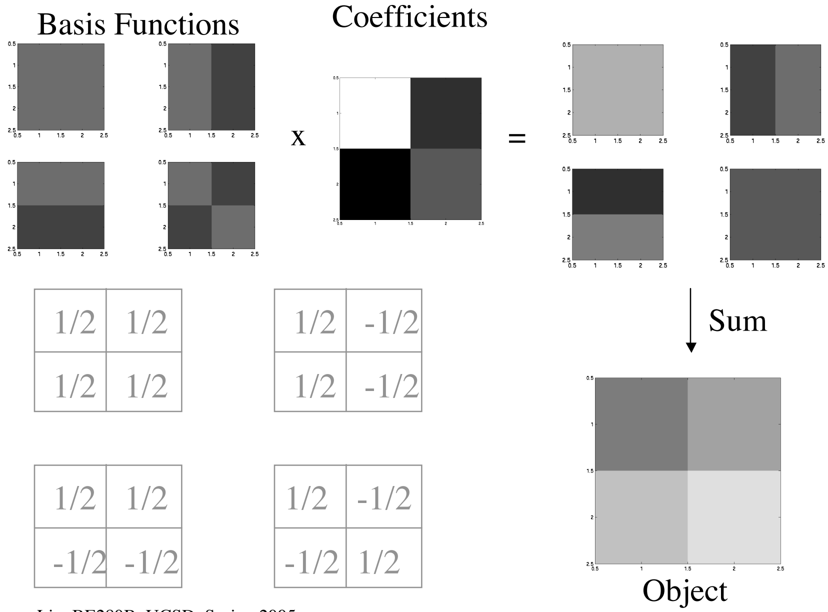
2D Signal

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Image Decomposition

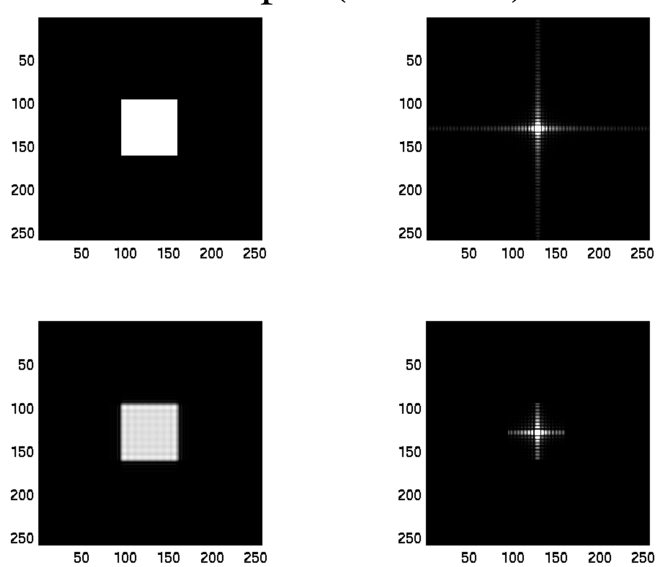
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} g[m,n] &= a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1] \\ &= \sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l] \\ &= \sum_{k=0}^{1}\sum_{l=0}^{1} c_{k,l} b_{k,l}[m,n] \end{aligned}$$

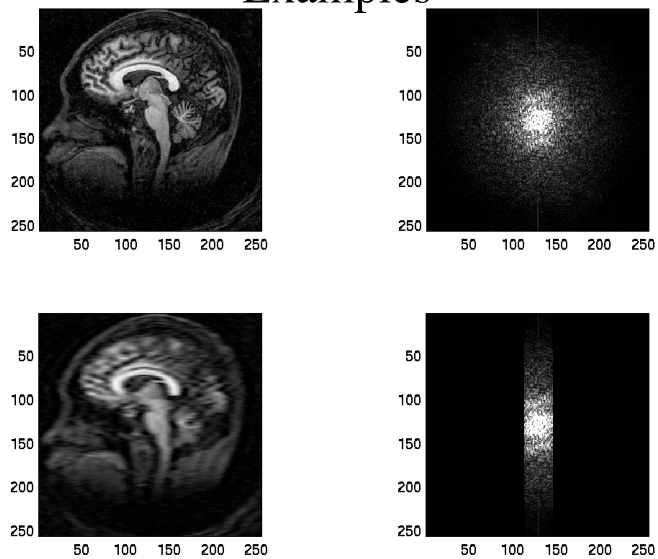


Thomas Liu, BE280B, UCSD, Spring 2005

Example (sinc/rect)



Examples



Signal Expansions

1D Discrete-Time Series Expansion

$$y[n] = \sum_{i=-\infty}^{\infty} c_i b_i[n]$$

2D discrete expansion is

$$y[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l} b_{k,l}[m,n]$$

Imaging and Basis Functions

1. Most imaging methods may be considered to be the process of taking the inner product of an object with a set of basis functions, where the basis functions are determined by physics and engineering. In other words, the basis functions act as our “rulers” for measuring the object.
2. Fourier bases show up frequently because the world is full of harmonic oscillators, e.g. MRI.
3. The basis functions are not necessarily orthogonal.
4. In fact, the “basis” functions usually do not even form a complete basis, so that the best we can do is approximate the original object given our measurements.

Inner Products

$$\langle r,s\rangle = \begin{cases} \sum_{n=1}^{N} r^*[n]s[n] & \text{for finite-dimensional vectors} \\ \sum_{n=-\infty}^{\infty} r^*[n]s[n] & \text{for infinite-dimensional vectors} \\ \int_{t=-\infty}^{\infty} r^*(t)s(t)dt & \text{for continuous signals} \end{cases}$$

The norm is defined as

$$\|s\| = \sqrt{\langle s,s\rangle}$$

Orthogonality

Some other notations for the inner product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \bullet \mathbf{y} = \mathbf{x}^T \mathbf{y}$$

Also, recall that the angle between the two vectors is given by

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

Two vectors are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, and therefore $\theta = \pi/2$.

Orthonormal basis

A set of vectors $S = \{\mathbf{b}_i\}$ forms an orthonormal basis, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for $i \neq j$, every basis vector is normalized to have unit length $\|\mathbf{b}_i\| = 1$, and any vector $\mathbf{y}$ in the space can be expressed as a linear combination of the basis vectors, i.e. $\mathbf{y} = \sum_k c_k \mathbf{b}_k$.

Examples: Fourier basis, Wavelet basis, Hadamard basis

Finding Expansion Coefficients

Define the basis matrix as $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \end{bmatrix}$.

Then any vector $\mathbf{y} = \mathbf{B}\mathbf{c} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

Multiply both sides of the equation by $\mathbf{B}^{-1}$, to obtain $\mathbf{c} = \mathbf{B}^{-1}\mathbf{y}$.
Because the basis vectors are orthonormal $\mathbf{B}^H\mathbf{B} = \mathbf{I}$, and therefore $\mathbf{B}^{-1} = \mathbf{B}^H$. So, we can also write $\mathbf{c} = \mathbf{B}^H\mathbf{y}$.
By definition, **B** is an orthonormal or unitary matrix.

Expansion Coefficients

$$\mathbf{c} = \mathbf{B}^H\mathbf{y} = \begin{bmatrix} \mathbf{b}_1^H \\ \mathbf{b}_2^H \\ \vdots \\ \mathbf{b}_N^H \end{bmatrix} \mathbf{y} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{y} \rangle \\ \langle \mathbf{b}_2, \mathbf{y} \rangle \\ \vdots \\ \langle \mathbf{b}_N, \mathbf{y} \rangle \end{bmatrix}$$

For any vector **y,** the *i*th expansion coefficient is the inner product of the *i*th orthonormal basis vector with **y**.

Orthonormal Signal Expansions

1D Discrete-Time Series Expansion

$$y[n] = \sum_{i=-\infty}^{\infty} c_i b_i\left[n\right] \qquad c_i = \left\langle b_i[n], y[n] \right\rangle$$

2D discrete expansion is

$$y[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{k,l} b_{k,l}[m,n] \quad c_{k,l} = \left\langle b_{k,l}[m,n], y[m,n] \right\rangle$$